Nesin Mathematics Village - Summer School 2026

The Geometry of Shapes: An Introduction to Algebraic Surfaces

Özgür İnce
Sivas Cumhuriyet University

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Disclaimer

Any errors or omissions in these lecture notes are the sole responsibility of the author. If you notice any mistakes or have suggestions for improvement, please kindly report them to:

ozgince@cumhuriyet.edu.tr

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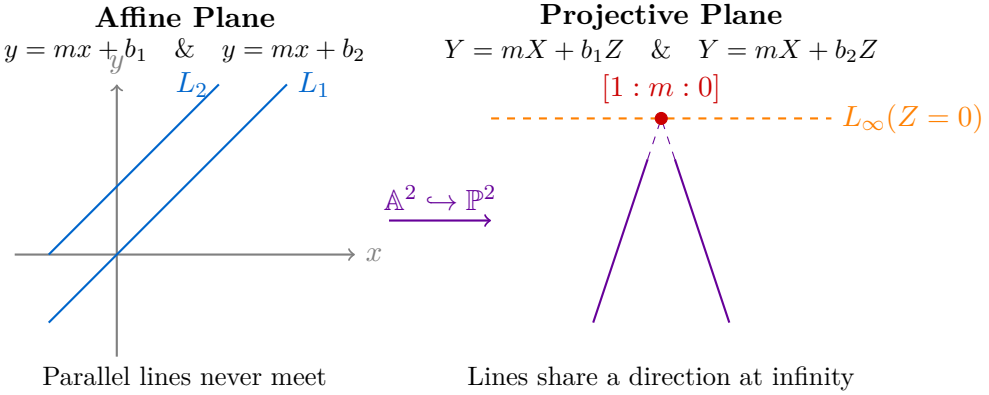
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Chapter 1

From the Projective Plane to Surfaces

1.1 Introduction: Why Projective Space?

Algebraic geometry studies the solution sets of polynomial equations. In affine geometry, intersections may disappear at infinity. For example, two distinct parallel affine lines do not meet. After passing to projective space, these lines acquire a common point at infinity. Together with working over an algebraically closed field and counting intersection multiplicities, this allows intersection theorems such as Bézout's theorem to take their cleanest form.



Parallel lines never meet

Lines share a direction at infinity

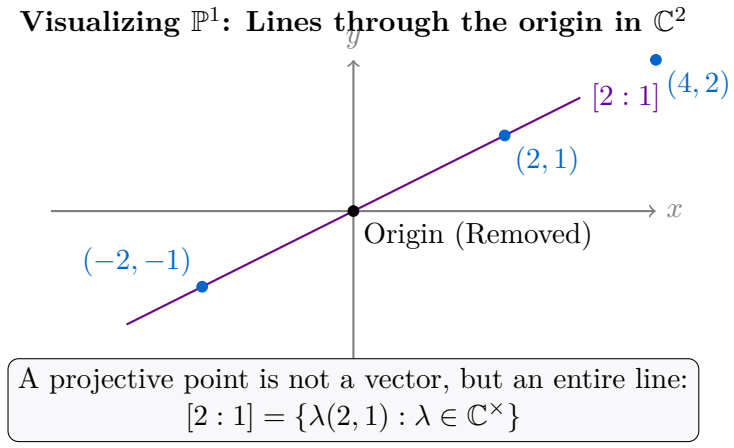
1.2 Projective Space and Points at Infinity

1.2.1 Mathematical Definition

Geometrically, the complex projective n-space $\mathbb{P}^n$ is the set of complex lines through the origin in $\mathbb{C}^{n+1}$. Formally, we define $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where the equivalence relation is given by:

$$(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n), \quad \lambda \in \mathbb{C}^\times$$

We denote these equivalence classes by $[x_0 : x_1 : \cdots : x_n]$ and call them *homogeneous coordinates*.

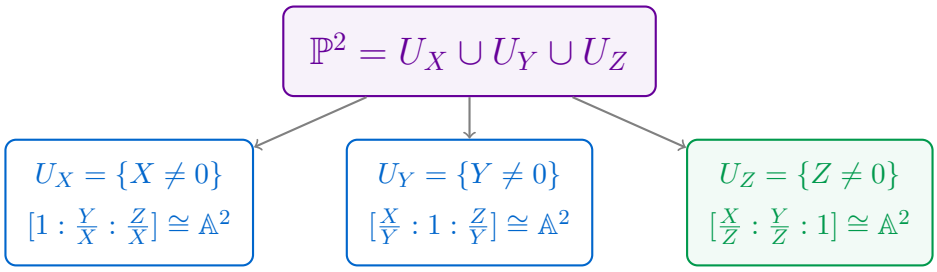


Remark 1.1 (Equivalence Classes)**.** *It is crucial to remember that* $[x_0 : \cdots : x_n]$ *is an equivalence class, not a vector. For instance,* $[1:2:3] = [2:4:6]$ *and* $[1:2:3] = [-1:-2:-3]$*, but* $[1:2:3] \neq [1:2:4]$.
Note: The diagram above shows the real slice of the corresponding complex line.

1.2.2 Embedding Affine Space into Projective Space

Let's consider $\mathbb{P}^2$ (the projective plane) with coordinates $[x:y:z]$. We can define the subset $U_z = \{[x:y:z] \in \mathbb{P}^2 \mid z \neq 0\}$. Since $z \neq 0$, we can divide by z to write any point in this set uniquely as $[x/z : y/z : 1]$. This gives an isomorphism $U_z \cong \mathbb{A}^2$, embedding the affine plane into the projective plane via $(u, v) \mapsto [u:v:1]$.

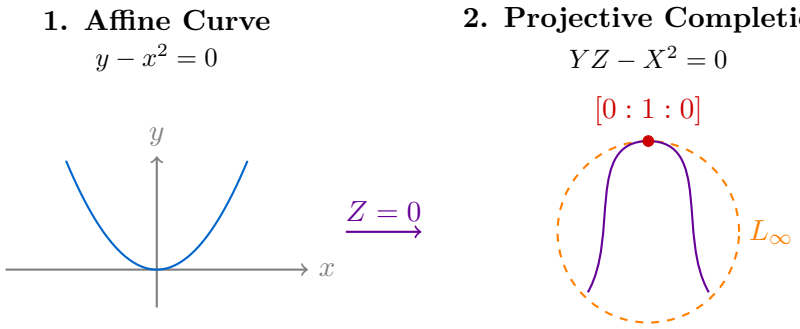
The points not in U_z are those where $z = 0$, forming the **line at infinity** (L_∞). Thus, $\mathbb{P}^2 = U_z \cup L_\infty$. It is important to note that $\mathbb{P}^2$ is not a mysterious new space, but an atlas covered by three such affine charts:



1.2.3 Homogenization

How do we generally "pass to projective space"? If $f(x, y)$ has total degree d, its homogenization is given by:

$$F(X, Y, Z) = Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$$



Both directions along the real affine parabola approach the same projective point at infinity. Notice that the projective parabola is tangent to the line at infinity L_∞ at $[0 : 1 : 0]$; we will explore this intersection multiplicity (which is 2) in the next lecture on Bézout's Theorem!

1.3 Algebraic Surfaces and Degree

In this course, our first examples of algebraic surfaces will be *projective hypersurfaces* in $\mathbb{P}^3$.

Definition 1.2. *A projective hypersurface of degree d is the zero locus*

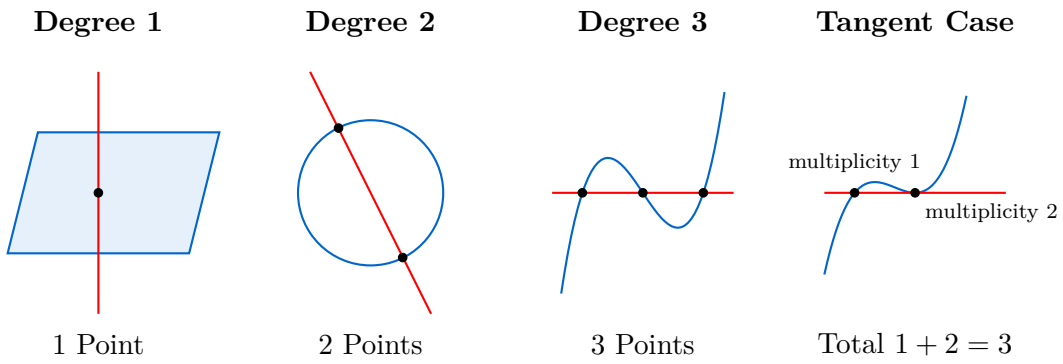
$$S = V(F) \subset \mathbb{P}^3,$$

where $F \in \mathbb{C}[x, y, z, w]$ is a nonconstant homogeneous polynomial of degree $d \geq 1$.

Why homogeneous? If F is homogeneous of degree d, then $F(\lambda x, \lambda y, \lambda z, \lambda w) = \lambda^d F(x, y, z, w)$. Thus, the condition $F(x, y, z, w) = 0$ is independent of the choice of representative for the equivalence class, making the zero locus well-defined.

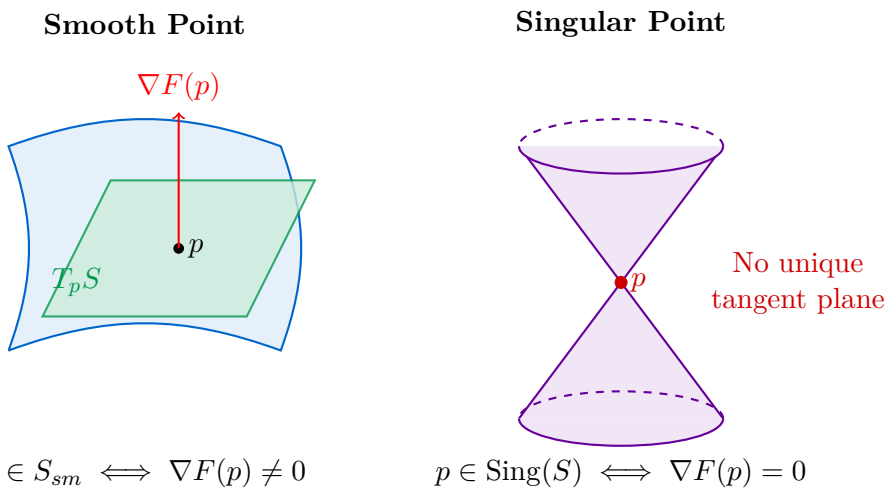
The Geometric Meaning of Degree: If $S = V(F) \subset \mathbb{P}^3$ is a surface of degree d, then the restriction of F to a line $L \not\subset S$ is a homogeneous polynomial of degree d in two variables. Hence, $L \cap S$ has a total intersection multiplicity of exactly d.

Note: The diagrams below are schematic plane sections of surfaces by a general line.



1.4 Smooth Surfaces and Singular Points

At a *smooth* point, the linear part of the local equation determines a unique tangent plane. At a *singular* point, all first-order terms vanish, so the equation does not determine a tangent plane in the expected way.



Theorem 1.3 (Jacobian Criterion)**.** *Let $S = V(F) \subset \mathbb{P}^3$ be an algebraic surface, where F is square-free (i.e., F has no repeated irreducible factors). A point $p = [a : b : c : d] \in \mathbb{P}^3$ is a* ***singular point*** *of S if and only if:*

$$F(p) = 0 \quad \textit{and} \quad \frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = \frac{\partial F}{\partial w}(p) = 0$$

The square-free assumption is essential: for $F = x^2$, the zero locus $V(F)$ is the smooth plane $x = 0$, yet all partial derivatives of F vanish along it.

We denote the set of all singular points as $\mathrm{Sing}(S) = V(F, F_x, F_y, F_z, F_w)$. The smooth locus is $S_{sm} = S \setminus \mathrm{Sing}(S)$.

Remark 1.4 (Euler's Formula)**.** *For a homogeneous polynomial F of degree d, Euler's formula states:*

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} + w\frac{\partial F}{\partial w} = d \cdot F$$

Since we work over $\mathbb{C}$ (characteristic 0), if all partial derivatives vanish at p, then $F(p) = 0$ is automatically satisfied.

If p is a smooth point of S, the unique tangent plane T_pS is given by the equation below. We use $[X : Y : Z : W]$ as variable coordinates on the ambient projective space, while $p = [a : b : c : d]$ is the fixed point of tangency.

$$T_pS = V\left(\frac{\partial F}{\partial x}(p)X + \frac{\partial F}{\partial y}(p)Y + \frac{\partial F}{\partial z}(p)Z + \frac{\partial F}{\partial w}(p)W\right)$$

1.4.1 Examples

1. Smooth Surface (The Fermat Cubic Surface):
$F(x, y, z, w) = x^3 + y^3 + z^3 + w^3 = 0$. The gradient is $\nabla F = (3x^2, 3y^2, 3z^2, 3w^2)$. For all derivatives to vanish, we need $x = y = z = w = 0$. Since this is not a valid point in $\mathbb{P}^3$, the surface is everywhere smooth.

2. Singular Surface (The Quadric Cone):
$F(x, y, z, w) = x^2 + y^2 - z^2 = 0$. The partial derivatives are $F_x = 2x$, $F_y = 2y$, $F_z = -2z$, $F_w = 0$. They all vanish only at $p = [0 : 0 : 0 : 1]$. This equation describes the projective cone over the smooth conic $x^2 + y^2 - z^2 = 0$ in the plane $w = 0$, with its vertex at $[0 : 0 : 0 : 1]$.

Problem Session: Finding Singularities & Homogenization

Note: The surfaces mentioned are complex projective surfaces; visualizing them often relies on real affine sections.

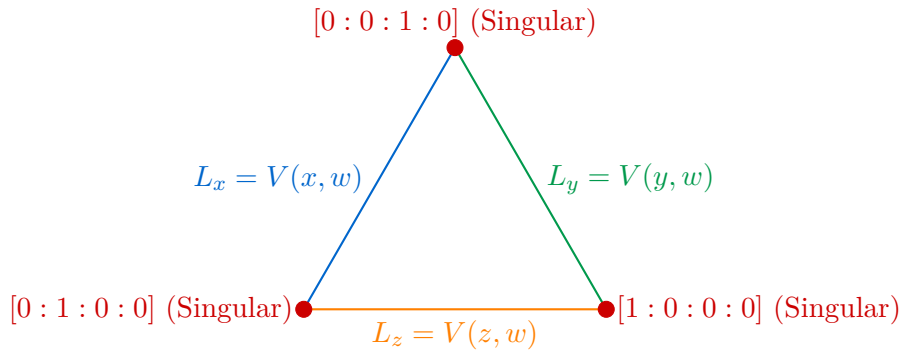
1. Find the intersection of the degree 3 surface $S = V(xyz - w^3) \subset \mathbb{P}^3$ with the plane at infinity $w = 0$. Then, calculate the coordinates of its singular points.

2. Homogenize the affine curve $y = x^2$ and find its points at infinity.

3. Find the tangent plane to the Fermat cubic surface $V(x^3 + y^3 + z^3 + w^3)$ at the point $p = [1 : -1 : 0 : 0]$.

Solutions to Problem Session

Problem 1.
At infinity $w = 0$, the equation $xyz - w^3 = 0$ reduces to $xyz = 0$. The intersection is the union of three projective lines in $\mathbb{P}^2$: $L_x = V(x, w), L_y = V(y, w), L_z = V(z, w)$.



$$S \cap H_\infty = L_x \cup L_y \cup L_z$$
$$\mathrm{Sing}(S) = (L_x \cap L_y) \cup (L_y \cap L_z) \cup (L_z \cap L_x)$$

For singular points, we compute the partial derivatives: $F_x = yz$, $F_y = xz$, $F_z = xy$, and $F_w = -3w^2$. Setting them to zero yields $w = 0$ and at least two of x, y, z must be zero. The singular points are exactly the intersection points of the three lines at infinity:

$$\mathrm{Sing}(S) = \{[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0]\}.$$

Problem 2.
The affine equation is $f(x, y) = y - x^2 = 0$. Since the degree is $d = 2$, the homogenization is:

$$F(X, Y, Z) = Z^2 \left(\frac{Y}{Z} - \frac{X^2}{Z^2} \right) = YZ - X^2 = 0$$

To find the points at infinity, we set $Z = 0$. This gives $-X^2 = 0 \implies X = 0$. Thus, there is only one point at infinity: $[0 : 1 : 0]$.

Problem 3.
For $F = x^3+y^3+z^3+w^3$ and $p = [1 : -1 : 0 : 0]$, the gradient is $\nabla F(p) = (3x^2, 3y^2, 3z^2, 3w^2)|_p = (3, 3, 0, 0)$. Using the coordinates $[X : Y : Z : W]$ for the tangent plane equation, we get:

$$3X + 3Y + 0Z + 0W = 0 \implies X + Y = 0$$

Further Reading & Resources for Day 1

To deepen your understanding of today's topics, you may refer to the following resources:

- **Miles Reid**, *Undergraduate Algebraic Geometry*: Chapter 1 & 2 provide excellent intuition for the transition from affine to projective geometry, and the homogenization of conic sections.
- **Andreas Gathmann**, *Algebraic Geometry Class Notes*: A formal and rigorous treatment of algebraic sets (Chapters 1 & 2) and projective varieties (Chapters 6 & 7). Available online: https://agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf
- **Richard E. Borcherds**, *Algebraic Geometry I: Varieties* (YouTube Series): Highly recommended for a visual and comprehensive introduction to projective spaces. https://www.youtube.com/playlist?list=PL8yHsr3EFj53j51FG6wCbQKjBgpjKa5PX

Chapter 2

Curves and the Logic of Intersection

2.1 Introduction: Zooming in on the Surface

On our first day, we looked at algebraic surfaces as a whole, focusing on their defining equations in projective space and identifying their smooth or singular points. Today, we change our perspective. We will walk *on* the surface and study the 1-dimensional geometries living there: **Curves**.

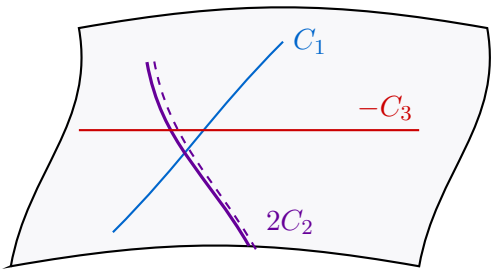
To understand a surface, one of our most powerful strategies is to study the curves lying on it and how they intersect. This intersection theory will become one of our main tools in the study and classification of surfaces.

2.2 Divisors: Curves with Multiplicity

A curve C on a surface S is a 1-dimensional subvariety. However, in intersection theory, simply knowing *which* curve we have is not enough; we also need to know *how many times* it appears. We may want to record the same curve more than once, just as a polynomial may have a root with multiplicity.

To handle this, we introduce the concept of a **Divisor**. A *Weil divisor* on a smooth surface is a finite formal integer linear combination of irreducible curves C_i:

$$D = \sum_{i=1}^{k} a_i C_i \quad (a_i \in \mathbb{Z})$$



A Divisor $D = C_1 + 2C_2 - C_3$ on a surface patch S

Negative coefficients (like $-C_3$) might seem strange geometrically, but they are algebraically essential when we deal with rational functions. The principal divisor associated with a rational function f is given by:

$$\mathrm{div}(f) = \sum_{C} \mathrm{ord}_C(f) C$$

where $\text{ord}_C(f) > 0$ indicates that f has a zero along C, and $\text{ord}_C(f) < 0$ indicates a pole along C.

2.3 The Intersection Number $(C \cdot D)$

Let $C \neq D$ be irreducible curves on a smooth projective surface. We define the **intersection number** $C \cdot D$ (or $C.D$) as the number of intersection points, *counted with multiplicity.*

For any point $p \in C \cap D$, we define the local intersection multiplicity $i_p(C, D)$.

Remark 2.1 (Local Intersection Multiplicity)**.** *For readers familiar with local rings, the precise algebraic definition is:*

$$i_p(C, D) = \dim_{\mathbb{C}} \mathcal{O}_{S,p}/(f, g)$$

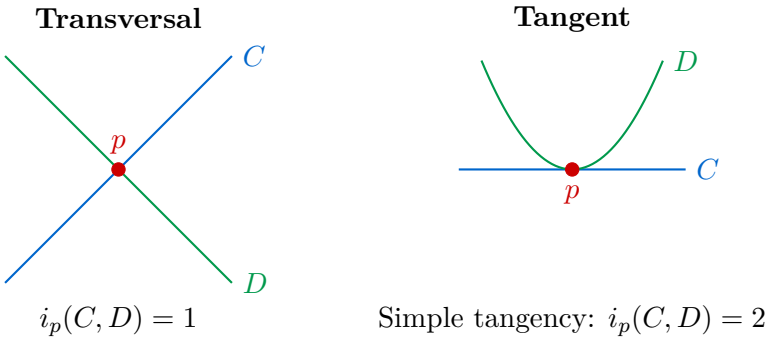
where f and g are the local equations defining C and D near p.

Geometrically:

- If C and D cross transversally (like an "X") at p, then $i_p(C, D) = 1$.
- If C and D are tangent at p, then $i_p(C, D) \geq 2$.

When C and D share no common irreducible components, the global intersection number is the sum over all intersection points:

$$C \cdot D = \sum_{p \in C \cap D} i_p(C, D)$$



$i_p(C, D) = 1$ Simple tangency: $i_p(C, D) = 2$

Properties of the Intersection Form: The intersection pairing has two beautiful properties that make calculations incredibly easy for any divisors:

1. **Symmetry:** $C \cdot D = D \cdot C$
2. **Bilinearity:** $(aC_1 + bC_2) \cdot D = a(C_1 \cdot D) + b(C_2 \cdot D)$

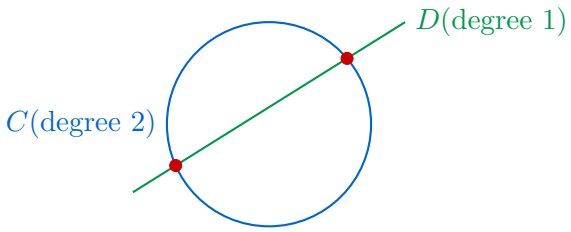
2.4 Bézout's Theorem (Intersection in $\mathbb{P}^2$)

Projective space prevents intersections from escaping to infinity. Over an algebraically closed field, and after counting multiplicities, Bézout's theorem takes its clean form in $\mathbb{P}^2$.

Theorem 2.2 (Bézout's Theorem)**.** *Let C and D be two projective curves in $\mathbb{P}^2$ of degrees m and n, respectively, sharing no common components. Then their total intersection number is exactly $m \times n$:*

$$C \cdot D = m \cdot n$$

Recall the parabola $y = x^2$ from Day 1. Its homogenization $YZ - X^2 = 0$ (degree 2) intersects the line at infinity $Z = 0$ (degree 1) at exactly one distinct point: $[0 : 1 : 0]$. However, substituting $Z = 0$ yields $X^2 = 0$, meaning the intersection multiplicity is 2. Bézout's theorem guarantees $2 \times 1 = 2$ intersection points counting multiplicities, perfectly matching our algebraic result!



A conic ($m = 2$) and a line ($n = 1$) intersect at $2 \times 1 = 2$ points.

2.5 Linear Equivalence

Two divisors D and D' are **linearly equivalent**, written $D \sim D'$, if their difference is the divisor of a nonzero rational function:

$$D - D' = \operatorname{div}(f)$$

When D and D' are effective, one may often visualize them as different members of the same linear system.

The most crucial rule of intersection theory is its behavior under this equivalence:

Invariance under Linear Equivalence

Intersection numbers are invariant under linear equivalence. If $C \sim C'$, then for any divisor E:

$$C \cdot E = C' \cdot E$$

The Moving Lemma: Under suitable conditions, a divisor can be replaced by a linearly equivalent divisor that meets a given curve properly, meaning that they share no common irreducible component. In favorable situations, the moved divisor may also be chosen to meet the curve transversally.

2.6 The Mystery of "Self-Intersection"

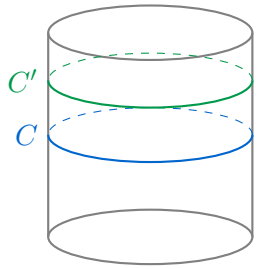
What if we want to intersect a curve C with itself? What does $C \cdot C$ (or C^2) mean? If we try to literally intersect $C \cap C$, they overlap entirely, sharing infinitely many points!

The self-intersection C^2 is defined by the intersection pairing on divisor classes. If the linear system of C contains another curve C' meeting C properly, then:

$$C^2 = C \cdot C'$$

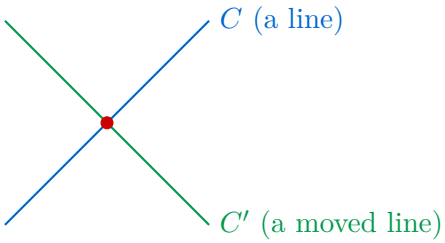
because $C \sim C'$. This gives a useful geometric way to compute self-intersection when C can move.

Case 1: $C^2 = 0$



C moves to a disjoint C'.
$C \cap C' = \emptyset \implies C^2 = 0$

Case 2: $C^2 > 0$



In $\mathbb{P}^2$, two lines always intersect.
$C \cdot C' = 1 \implies C^2 = 1$

What if a curve cannot move? When C cannot be moved, C^2 is still well-defined algebraically. Negative self-intersection is a strong sign of rigidity. In particular, an irreducible curve C with $C^2 < 0$ cannot move at all: it is the only effective divisor linearly equivalent to it, i.e. $|C| = \{C\}$. (Indeed, any other effective divisor $C' \sim C$ could not contain C, since otherwise $C' - C$ would be a nonzero effective divisor linearly equivalent to 0; hence $C \cdot C' \geq 0$, contradicting $C \cdot C' = C^2 < 0$.) We will encounter these strange objects tomorrow when we "blow up" points!

Remark 2.3 (Normal Bundle)**.** *The normal bundle measures how the curve sits inside the surface. Geometrically, if C is a smooth curve on a smooth surface S, its self-intersection is the degree of its normal bundle $\mathcal{N}_{C/S}$:*

$$C^2 = \deg \mathcal{N}_{C/S}$$

Problem Session: Bézout, Bilinearity, and Self-Intersection

1. Let C be a cubic curve (degree 3) and D be a quartic curve (degree 4) in $\mathbb{P}^2$. Assuming they intersect transversally at every point, exactly how many points lie in $C \cap D$?
2. On a surface S, we have three curves C_1, C_2, C_3. Their intersection numbers are known: $C_1 \cdot C_2 = 1$, $C_1 \cdot C_3 = 0$, $C_2 \cdot C_3 = 1$. Furthermore, $C_1^2 = -1$ and $C_2^2 = 0$. Calculate the self-intersection of the divisor $D = 2C_1 - C_2 + C_3$. (Assume $C_3^2 = 1$).
3. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, which is isomorphic to a smooth quadric surface in $\mathbb{P}^3$. Let H and V denote fibers of the two projections. A horizontal fiber H and a vertical fiber V meet transversally in exactly one point. What is H^2 and V^2? What is $(H + V)^2$?

Solutions to Problem Session

Problem 1.
Since C and D are curves in the projective plane $\mathbb{P}^2$, we can directly apply Bézout's Theorem. The number of intersection points (counting multiplicity) is the product of their degrees.

$$C \cdot D = \deg(C) \times \deg(D) = 3 \times 4 = 12$$

Since they intersect transversally everywhere (meaning all local intersection multiplicities $i_p = 1$), there are exactly 12 distinct intersection points.

Problem 2.
First, let's write the general algebraic expansion for the self-intersection of a linear combination:

$$(aC_1 + bC_2 + cC_3)^2 = a^2C_1^2 + b^2C_2^2 + c^2C_3^2 + 2ab(C_1 \cdot C_2) + 2ac(C_1 \cdot C_3) + 2bc(C_2 \cdot C_3)$$

Applying this bilinearity and symmetry to $D^2 = (2C_1 - C_2 + C_3)^2$:

$$\begin{aligned} D^2 &= 4C_1^2 + C_2^2 + C_3^2 - 4(C_1 \cdot C_2) + 4(C_1 \cdot C_3) - 2(C_2 \cdot C_3) \\ &= 4(-1) + 0 + 1 - 4(1) + 4(0) - 2(1) \\ &= -4 + 1 - 4 - 2 \\ &= -9 \end{aligned}$$

Remark 2.4 (Linear Algebra Viewpoint)**.** *We can elegantly compute this using an intersection matrix M and a coefficient vector v:*

$$M = \begin{pmatrix} C_1^2 & C_1 \cdot C_2 & C_1 \cdot C_3 \\ C_1 \cdot C_2 & C_2^2 & C_2 \cdot C_3 \\ C_1 \cdot C_3 & C_2 \cdot C_3 & C_3^2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The self-intersection is a quadratic form: $D^2 = v^T M v = -9$.

Remark 2.5 (A Geometric Model)**.** *These numbers are realized by an actual surface, which we will meet tomorrow: the blow-up of $\mathbb{P}^2$ at a point p. Take $C_1 = E$ (the exceptional curve), C_2 the strict transform of a line through p, and C_3 a line not passing through p. Writing H for the class of a line, their classes are $C_1 = E$, $C_2 = H - E$, $C_3 = H$, so*

$$D = 2E - (H - E) + H = 3E \quad \textit{and} \quad D^2 = 9E^2 = -9,$$

in agreement with the computation above.

Problem 3.
On $S = \mathbb{P}^1 \times \mathbb{P}^1$:

- **Calculating H^2:** Distinct fibers of the same projection are disjoint. Since $H \cap H' = \emptyset$, we have $H \cdot H' = 0$. Thus, $H^2 = 0$.
- **Calculating V^2:** By the exact same logic (distinct vertical fibers are disjoint), $V \cap V' = \emptyset$. Thus, $V^2 = 0$.
- **Calculating $(H+V)^2$:** We are given that a horizontal and vertical fiber intersect transversally exactly once, so $H \cdot V = 1$. Using bilinearity:

$$(H + V)^2 = H^2 + 2(H \cdot V) + V^2 = 0 + 2(1) + 0 = 2$$

Further Reading & Resources for Day 2

To deepen your understanding of today's topics, you may refer to the following resources:

- **Miles Reid**, *Undergraduate Algebraic Geometry*: Chapter I, §1 (Plane conics) introduces intersections of plane curves and an elementary form of Bézout's Theorem.
- **Arnaud Beauville**, *Complex Algebraic Surfaces*: Chapter 1 provides a very formal definition of Divisors, Linear Equivalence, and the Intersection Form on surfaces.
- **Richard E. Borcherds**, *Algebraic Geometry I: Varieties* (YouTube Series): Excellent video lectures covering Divisors and Intersection Theory conceptually. https://www.youtube.com/playlist?list=PL8yHsr3EFj53j51FG6wCbQKjBgpjKa5PX

Chapter 3

The Blow-up Operation

3.1 Introduction: The Microscope of Geometry

Sometimes, multiple curves intersect at a single point, creating a tangled, complicated local geometry. Or worse, a surface itself might have a singular point (a sharp peak or self-intersection) that prevents us from using smooth geometry.

To resolve these issues, algebraic geometers use an operation called a **Blow-up**. You can think of a blow-up as a "mathematical microscope." It replaces a single tangled point with an entire projective line, spreading out all the different directions that approach that point.

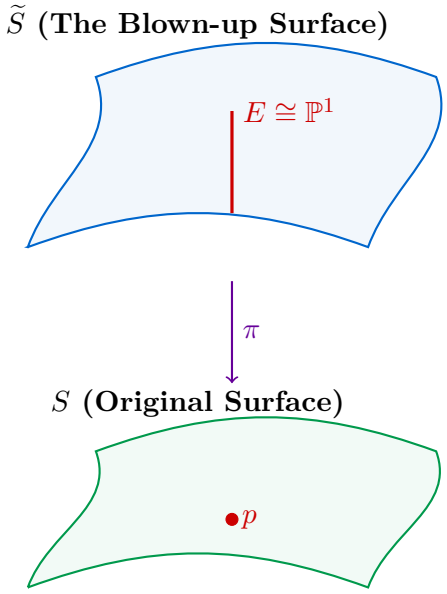
Remark 3.1. *A single blow-up does not necessarily resolve every singularity. In general, a sequence of carefully chosen blow-ups may be required to fully smooth out a tangled point.*

3.2 The Geometry of a Blow-up

Let p be a smooth point on a surface S. The blow-up of S at p creates a new surface $\widetilde{S}$ and a projection map:

$$\pi : \widetilde{S} \to S$$

This map has two crucial properties:
1. Everywhere except at p, the map is a perfect one-to-one correspondence (an isomorphism). $\widetilde{S} \setminus \pi^{-1}(p) \cong S \setminus \{p\}$.
2. The point p itself is replaced by an entire projective line $\mathbb{P}^1$. This line is called the **Exceptional Divisor** and is denoted by $E = \pi^{-1}(p)$.



Every point on the exceptional divisor E corresponds to a unique *tangent direction* (slope) passing through p on the original surface S.

3.2.1 A Simple Example: Blowing up the Origin

Let's see this in action. Consider the affine plane $\mathbb{A}^2$ with coordinates (x, y) and blow up the origin $p = (0, 0)$. The blown-up surface $\widetilde{\mathbb{A}^2}$ is defined in $\mathbb{A}^2 \times \mathbb{P}^1$ by the equation:

$$xv = yu$$

where $[u : v]$ are the homogeneous coordinates of $\mathbb{P}^1$.

If $(x, y) \neq (0, 0)$, then the ratio $[u : v]$ is completely forced to be $[x : y]$. Thus, away from the origin, nothing changes! But if $(x, y) = (0, 0)$, the equation $0 \cdot v = 0 \cdot u$ is satisfied for **any** $[u : v]$. Thus, the preimage of the origin is exactly $\{0\} \times \mathbb{P}^1$, which is our exceptional line E.

To do practical calculations, we cover $\widetilde{\mathbb{A}^2}$ with two affine charts:

- **Chart 1 ($u \neq 0$):** Let $t = v/u$. The equation $xv = yu$ becomes $x(tu) = yu \implies y = xt$. The local coordinates are (x, t), and the exceptional divisor E (where $x = y = 0$) is given by $x = 0$.
- **Chart 2 ($v \neq 0$):** Let $s = u/v$. The equation becomes $x = ys$. The local coordinates are (s, y), and E is given by $y = 0$.

This explicit coordinate description proves that the blown-up surface is still perfectly smooth everywhere (covered by two $\mathbb{A}^2$ charts)! It also gives us a powerful algebraic way to compute strict transforms by simply factoring out the exceptional divisor.

3.3 The Art of Separating Intersecting Curves

Why replace a single point with a whole line? Imagine two curves crossing at p. When we blow up p, the curves are pulled up to $\widetilde{S}$. Since they pass through p with different slopes, they will hit the exceptional line E at **different points**!

Let C be a curve on S passing through p.

- The **Total Transform** π^*C is the entire preimage of C. It includes the strict transform and a multiple of E: $\pi^*C = \widetilde{C} + m_p(C)E$, where $m_p(C)$ is the multiplicity of p on C.

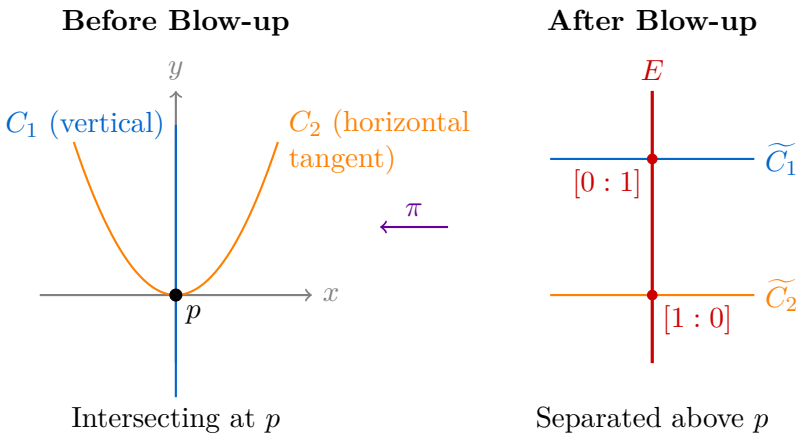
- The **Strict Transform** $\widetilde{C}$ is the true "lifted" curve, ignoring the exceptional line E.

Example 3.2 (Simple Example: Separating Intersecting Curves)**.** *Let C_1 be the line $x = 0$ and C_2 be the parabola $y = x^2$. They intersect only at the origin $p = (0, 0)$. Let's calculate their strict transforms algebraically:*

- *For the parabola C_2 $(y = x^2)$, we use Chart 1 $(y = xt)$. Substituting gives $xt = x^2 \implies x(t - x) = 0$. The factor $x = 0$ is the exceptional divisor E. The remaining factor $t = x$ is the **strict transform** $\widetilde{C_2}$. It intersects E $(x = 0)$ at $t = 0$, which directly corresponds to the direction $[u : v] = [1 : 0]$.*
- *For the vertical line C_1 $(x = 0)$, we use Chart 2 $(x = ys)$. Substituting gives $ys = 0$. Since $y = 0$ is E, the strict transform $\widetilde{C_1}$ is $s = 0$. It intersects E at $s = 0$, which is the direction $[u : v] = [0 : 1]$.*

Even though they met at p downstairs, their strict transforms meet E at different points, so the intersection at the origin has been separated.

A blow-up separates branches with *distinct* tangent directions. Branches with the same tangent direction may still meet after the first blow-up, so further blow-ups may be necessary.



3.4 The Self-Intersection of E

Yesterday we saw that the self-intersection of a smooth curve is the degree of its normal bundle. For the exceptional divisor of a blow-up, $E \cong \mathbb{P}^1$ and its normal bundle is given by $\mathcal{N}_{E/\widetilde{S}} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$.

Therefore:

Theorem 3.3. *The self-intersection of the exceptional divisor E on a blown-up surface $\widetilde{S}$ is always exactly -1:*

$$E^2 = -1$$

*For this reason, exceptional divisors are often simply called -1-**curves**.*

This negative number does not count ordinary intersection points. It records how E is embedded in the blown-up surface.

Furthermore, we can calculate the intersections of strict transforms. Using the relations $\pi^*C = \widetilde{C} + m_p(C)E$, $\pi^*C \cdot E = 0$, $(\pi^*C)^2 = C^2$, and $E^2 = -1$, we derive two crucial formulas:

$$\widetilde{C} \cdot E = m_p(C)$$

$$\widetilde{C}^2 = C^2 - (m_p(C))^2$$

If a curve passes smoothly through p (multiplicity 1), then blowing up that point drops its self-intersection by exactly 1.

3.5 Minimal Surfaces: The Simplest Form

We have seen that blowing up a point creates a -1-curve. What if we run this process in reverse? Can we take a surface containing a -1-curve and "collapse" or "blow down" that curve to a single smooth point?

Yes! This is one of the most celebrated theorems in surface geometry.

Castelnuovo's Contractibility Criterion

If a smooth surface contains a smooth rational curve E (a curve isomorphic to $\mathbb{P}^1$) with self-intersection $E^2 = -1$, then E can be smoothly blown down. That is, there exists a surface S and a blow-up map $\pi : \widetilde{S} \to S$ such that E is the exceptional divisor.

From the viewpoint of birational classification, a surface containing a -1-curve is not yet minimal. We blow it down. We keep blowing down -1-curves until none are left.

Definition 3.4. *A -1-**curve** is a smooth rational curve $E \cong \mathbb{P}^1$ with $E^2 = -1$. A surface is called a **Minimal Surface** if it does not contain any -1-curves.*

Note that the condition $E \cong \mathbb{P}^1$ is essential: a curve of higher genus may also have self-intersection -1, but it cannot be blown down to a smooth point.

Every smooth projective surface over $\mathbb{C}$ admits a minimal model obtained by contracting finitely many -1-curves. Understanding these minimal surfaces is the ultimate goal of the Kodaira Classification, which we will explore on Day 6!

Problem Session: Transformations and Intersections

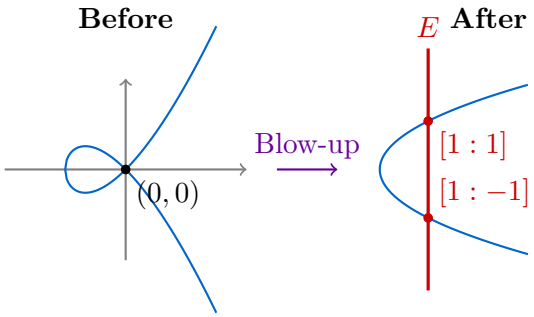
1. Consider the nodal cubic curve C given by the affine equation $y^2 = x^3 + x^2$. It crosses itself at the origin $(0, 0)$. What are the slopes of its two tangent lines at the origin? At which points $[u : v]$ will its strict transform $\widetilde{C}$ intersect the exceptional divisor E?

2. Let C be a smooth curve passing through p with multiplicity 1. Its total transform is $\pi^*C = \widetilde{C} + E$. Using the properties $E^2 = -1$ and $\widetilde{C} \cdot E = 1$, calculate the self-intersection of the strict transform $(\widetilde{C})^2$ in terms of C^2.

3. Suppose you blow up two distinct points p and q on $\mathbb{P}^2$ to get a new surface S. Let L be the line passing through p and q. What is the self-intersection of the strict transform $\widetilde{L}$ on S? Is it a -1-curve?

Solutions to Problem Session

Problem 1.
The equation is $y^2 = x^3 + x^2$. To find the tangents at the origin, we look at the lowest degree terms, which are $y^2 = x^2$. This factors as $(y - x)(y + x) = 0$. Thus, the curve has two tangent lines at the origin: $y = x$ (slope 1) and $y = -x$ (slope -1).
When we blow up the origin, the two branches of the curve will hit the exceptional divisor E at two distinct points corresponding to these slopes: $[1 : 1]$ and $[1 : -1]$. The node is successfully untangled!



Problem 2.
We are given the total transform $\pi^*C = \widetilde{C} + E$. A fundamental property of pullbacks is that they preserve intersection numbers from the base surface. So, $(\pi^*C)^2 = C^2$. Let's expand the left side using bilinearity:

$$\begin{aligned} C^2 &= (\widetilde{C} + E)^2 \\ &= (\widetilde{C})^2 + 2(\widetilde{C} \cdot E) + E^2 \end{aligned}$$

We substitute the known values $\widetilde{C} \cdot E = 1$ and $E^2 = -1$:

$$\begin{aligned} C^2 &= (\widetilde{C})^2 + 2(1) + (-1) \\ C^2 &= (\widetilde{C})^2 + 1 \\ (\widetilde{C})^2 &= C^2 - 1 \end{aligned}$$

Conclusion: Blowing up a point on a curve drops its self-intersection by exactly 1! (In the text, we derived the general formula $\widetilde{C}^2 = C^2 - (m_p(C))^2$. Since C is smooth at p, $m_p(C) = 1$, confirming our result).

Problem 3.
Let L be a line in $\mathbb{P}^2$. From Day 2, we know that two lines in $\mathbb{P}^2$ intersect exactly once, so $L^2 = 1$. The line L passes through p and q smoothly (multiplicity 1 for each). Using the formula derived in Problem 2, blowing up a point drops the self-intersection of the strict transform by 1. Since we blow up *two* distinct points (p and q) that lie on L:

$$(\widetilde{L})^2 = L^2 - 1 - 1 = 1 - 2 = -1$$

Yes! The strict transform $\widetilde{L}$ on the new surface S has a self-intersection of -1. According to Castelnuovo's criterion, $\widetilde{L}$ is now a -1-curve itself, and we could blow it down to create an entirely different surface!

Further Reading & Resources for Day 3

To deepen your understanding of today's topics, you may refer to the following resources:

- **Andreas Gathmann**, *Algebraic Geometry Class Notes*: Chapter 9 (Birational Maps and Blowing Up) gives a rigorous construction of blow-ups. Available online: https://agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf
- **Arnaud Beauville**, *Complex Algebraic Surfaces*: Chapter 2 offers a detailed mathematical dive into Blow-ups, Exceptional Divisors, and Castelnuovo's contractibility criterion for minimal surfaces.
- **Richard E. Borcherds**, *Algebraic Geometry I: Varieties* (YouTube Series): Watch the episodes specifically dedicated to "Blowing up" to visualize the process in multiple dimensions. https://www.youtube.com/playlist?list=PL8yHsr3EFj53j51FG6wCbQKjBgpjKa5PX

Chapter 4

Fingerprints of a Surface: Invariants

4.1 Introduction: What is an Invariant?

Imagine you are given two algebraic surfaces defined by completely different equations in different projective spaces. How can you tell if they are secretly the "same" surface?

To answer this, algebraic geometers look for **invariants**—mathematical fingerprints that remain unchanged under certain transformations. If two surfaces have different fingerprints, they cannot be the same. But remember a crucial principle in mathematics: *A difference in invariants guarantees two objects are different, but an equality of invariants does not guarantee they are the same.*

Today, we will explore topological invariants (which don't change if you bend or stretch the space) and birational invariants (which don't change even if you perform blow-ups!).

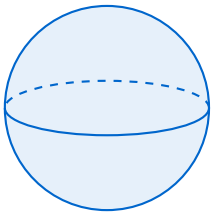
4.2 Real vs. Complex: The "Holes" of a Curve

Before we classify algebraic surfaces, we must resolve a classic point of confusion regarding dimensions.

- A **Complex Curve** has 1 complex dimension. But since $\mathbb{C} \cong \mathbb{R}^2$, it has **2 real dimensions**. Topologically, a smooth complex curve is a real 2D surface!
- A **Complex Surface** has 2 complex dimensions, meaning it has **4 real dimensions**.

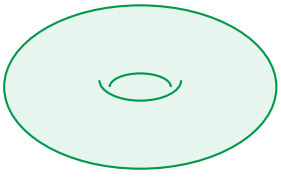
When people intuitively talk about the "number of holes of a surface," they are usually talking about the topology of a *complex curve* (also known as a Riemann Surface). The number of holes is called the **Genus** (g).

Genus $g = 0$



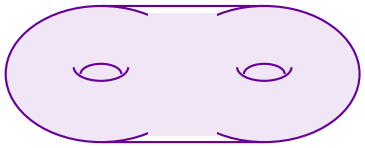
(e.g., Complex $\mathbb{P}^1$ or Conic)

Genus $g = 1$



(e.g., Smooth Cubic Curve)

Genus $g = 2$



(e.g., A Hyperelliptic Curve)

For a smooth curve C of degree d in the projective plane $\mathbb{P}^2$, we do not need to manually count the holes. Algebra gives us a magical formula:

Theorem 4.1 (The Degree-Genus Formula)**.**

$$g = \frac{(d-1)(d-2)}{2}$$

Note that no integer d yields $g = 2$ in this formula: a smooth curve of genus 2 is never isomorphic to a smooth plane curve. Any plane curve birational to it (for example, a plane quartic with a single node) must be singular.

4.3 Invariants of Algebraic Surfaces (Complex 2D)

Because a complex surface has 4 real dimensions, we cannot easily draw or visualize its "holes." Instead, we generalize the concept of genus using algebraic tools, specifically *differential forms* (things we can integrate).

For a smooth projective surface S, we define three fundamental numbers:

- **Geometric Genus (p_g):** Denoted algebraically as $p_g(S) = h^0(S, K_S) = h^{2,0}(S)$, this is the number of independent global holomorphic 2-forms on S. (Think of this as the number of independent "volume" measurements we can make).
- **Irregularity (q):** Denoted as $q(S) = h^1(S, \mathcal{O}_S) = h^{0,1}(S)$, this is the number of independent global holomorphic 1-forms on S.
- **Topological Euler Characteristic (χ_{top}):** An integer that alternates the counting of vertices, edges, faces, and higher-dimensional cells in a triangulation of the 4D real manifold.

We also define the **Holomorphic Euler Characteristic ($\chi(\mathcal{O}_S)$)**:

$$\chi(\mathcal{O}_S) = 1 - q + p_g$$

These numbers are tied together by the **Arithmetic Genus (p_a)**:

$$p_a = \chi(\mathcal{O}_S) - 1 = p_g - q$$

Example 4.2 (The Simplest Surface)**.** *For the projective plane $S = \mathbb{P}^2$:*

$$p_g = 0, \quad q = 0, \quad \Longrightarrow \quad p_a = 0$$

$$\chi_{top}(\mathbb{P}^2) = 3$$

If any surface has $p_g \neq 0$, we immediately know it cannot be isomorphic (or even birational) to $\mathbb{P}^2$!

4.4 Birational Invariants and Blow-ups

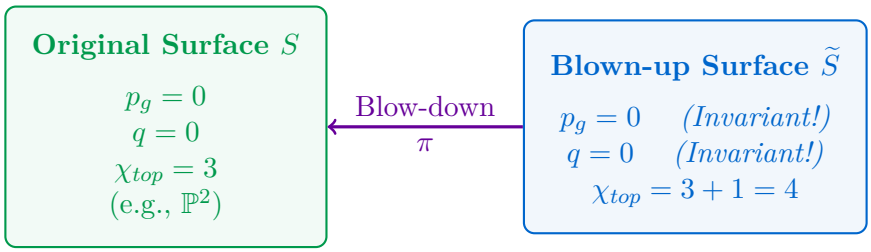
On Day 3, we learned how to blow up a surface to separate intersecting curves. A critical question arises: *Does a blow-up change the fingerprints of a surface?*

Surfaces connected by a sequence of blow-ups and blow-downs are called **birationally equivalent**.

Birational vs. Topological

- The geometric genus p_g and irregularity q **do not change** under blow-ups! They are *birational invariants.*
- The Euler characteristic χ_{top} **does change**. Since blowing up a point replaces it with an entire $\mathbb{P}^1$ (which is topologically a sphere with $\chi_{top} = 2$), the net change to the surface's Euler characteristic is:

$$\chi_{top}(\widetilde{S}) = \chi_{top}(S) + 1$$



This is why minimal models are central to birational classification. Blowing down a -1-curve decreases the topological Euler characteristic by 1. However, minimal models in the same birational class need not be unique; this happens precisely for rational and ruled surfaces.

Remark 4.3 (Looking Ahead: The Canonical Divisor)**.** *Another essential fingerprint is the self-intersection of the Canonical Divisor (K_S^2). Under a blow-up, $K_{\widetilde{S}}^2 = K_S^2 - 1$. We will use this when we study Cubic Surfaces on Day 5!*

Problem Session: Calculating Fingerprints

1. Consider a smooth curve C defined by a homogeneous polynomial of degree 1 (a line), degree 2 (a conic), and degree 3 (a cubic) in $\mathbb{P}^2$. What is the topological genus g for each of them? What shape does a smooth complex projective line have?
2. We know that for the projective plane, $\chi_{top}(\mathbb{P}^2) = 3$. Suppose we blow up two distinct points on $\mathbb{P}^2$ to create a new surface $\widetilde{S}$. What is $\chi_{top}(\widetilde{S})$?
3. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. We know that:

$$p_g(S) = q(S) = 0, \quad \chi_{top}(S) = 4.$$

Compare these values with those of $\mathbb{P}^2$ ($p_g = q = 0$, $\chi_{top} = 3$). Can these invariants alone determine whether the two surfaces are birationally equivalent? Is a blow-up of $\mathbb{P}^2$ at one point isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$?

Solutions to Problem Session

Problem 1.
We use the Degree-Genus formula $g = \frac{(d-1)(d-2)}{2}$:

- **Degree 1 (Line):** $g = \frac{(1-1)(1-2)}{2} = 0$. A complex projective line $\mathbb{P}^1(\mathbb{C})$ is topologically a sphere (the Riemann Sphere).
- **Degree 2 (Conic):** $g = \frac{(2-1)(2-2)}{2} = 0$. A smooth conic is also topologically a sphere!
- **Degree 3 (Cubic):** $g = \frac{(3-1)(3-2)}{2} = \frac{2\times1}{2} = 1$. A smooth cubic curve is an elliptic curve, and topologically, it is a **Torus** (a surface with exactly 1 hole).

Problem 2.
Blowing up a single point increases the topological Euler characteristic by exactly 1. Since we are performing 2 blow-ups on $\mathbb{P}^2$:

$$\chi_{top}(\widetilde{S}) = \chi_{top}(\mathbb{P}^2) + 1 + 1$$

$$\chi_{top}(\widetilde{S}) = 3 + 2 = 5$$

The new surface has an Euler characteristic of 5.

Problem 3.
Let's compare their fingerprints:

- For $\mathbb{P}^2$: $p_g = 0$, $q = 0$, $\chi_{top} = 3$.
- For $\mathbb{P}^1 \times \mathbb{P}^1$: $p_g = 0$, $q = 0$, $\chi_{top} = 4$.

No, these invariants alone cannot determine birational equivalence! Since p_g and q agree for both surfaces, these birational invariants provide no obstruction. However, their agreement does not prove birational equivalence. The topological Euler characteristic cannot decide the question because it naturally changes under blow-ups.

To prove they are birationally equivalent, we need a geometric argument: $\mathbb{P}^1 \times \mathbb{P}^1$ contains an affine open subset $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$. Similarly, $\mathbb{P}^2$ contains an open subset isomorphic to $\mathbb{A}^2$. Thus, both surfaces share the same function field $\mathbb{C}(x, y)$ and are therefore birational.

Are they isomorphic if we equalize their Euler characteristics? If we blow up a single point p on $\mathbb{P}^2$, we get $\chi_{top}(\mathrm{Bl}_p\mathbb{P}^2) = 4$. They now have the same Euler characteristic, but they are **not** isomorphic! $\mathrm{Bl}_p\mathbb{P}^2$ contains an exceptional -1-curve, while $\mathbb{P}^1 \times \mathbb{P}^1$ contains no -1-curves at all.

Although $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic, they are birational, and **both are minimal surfaces**! This proves a crucial point: minimal models are *not unique* in the birational class of rational surfaces. (In fact, uniqueness fails only for rational and ruled surfaces: every other surface, including the surfaces of "general type" we will meet on Day 6, has a unique minimal model.)
(Fascinating fact: The blow-up of $\mathbb{P}^2$ at two distinct points is isomorphic to the blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ at one point).

Further Reading & Resources for Day 4

To deepen your understanding of today's topics, you may refer to the following resources:

- **Miles Reid**, *Undergraduate Algebraic Geometry*: The appendix to Chapter I gives an excellent, intuitive discussion of the genus of a curve and its topology.
- **Arnaud Beauville**, *Complex Algebraic Surfaces*: Chapter 3 formally defines p_g, q, and χ, and proves their invariance under blow-ups.
- **Richard E. Borcherds**, *Algebraic Geometry I: Varieties* (YouTube Series): The lectures on differentials and Riemann-Roch beautifully explain how algebra captures topological "holes." https://www.youtube.com/playlist?list=PL8yHsr3EFj53j51FG6wCbQKjBgpjKa5PX

Chapter 5

The Magic of Cubic Surfaces: 27 Lines

5.1 Introduction: The Gem of Algebraic Geometry

If there is one theorem that captures the absolute magic of algebraic geometry, it is the discovery of the 27 lines on a cubic surface. Arthur Cayley and George Salmon discovered in 1849 that a smooth surface of degree 3 in $\mathbb{P}^3$ hides exactly 27 perfectly straight lines!

Today, we will use everything we have learned so far—degrees, blow-ups, and self-intersections—to uncover this beautiful truth.

5.2 Finding a Line: The Fermat Cubic

Let's start with a very easy, explicit example. A **Cubic Surface** $S \subset \mathbb{P}^3$ is defined by a homogeneous polynomial of degree 3. The most famous example is the **Fermat Cubic Surface**:

$$x^3 + y^3 + z^3 + w^3 = 0$$

Is there a straight line lying completely inside this curved surface? Recall that a line in $\mathbb{P}^3$ can be defined as the intersection of two planes. Let's try to set up two very simple planes:

$$x = -y \quad \text{and} \quad z = -w$$

If we substitute these into the Fermat equation:

$$(-y)^3 + y^3 + (-w)^3 + w^3 = -y^3 + y^3 - w^3 + w^3 = 0$$

It works perfectly! For any choice of y and w, the point $[-y : y : -w : w]$ is on the surface. We have found a straight line lying entirely on the cubic!

Remark 5.1 (More Lines)**.** *We can use complex numbers to find even more lines. Let ω be a complex number satisfying $\omega^3 = -1$, for example $\omega = e^{i\pi/3}$. The planes $x = \omega y$ and $z = \lambda w$ (where $\lambda^3 = -1$) will also form a line on this surface.*

5.3 The Theorem of Cayley and Salmon

What if the cubic surface is more complicated? What if it's completely generic and doesn't have such nice symmetries?

Theorem 5.2 (Cayley-Salmon, 1849)**.** *Every smooth cubic surface $S \subset \mathbb{P}^3$ over the complex numbers $\mathbb{C}$ contains exactly 27 straight lines.*

Why exactly 27? We can understand this beautifully by stepping away from the 3D space $\mathbb{P}^3$ and looking at the surface through the lens of a **Blow-up**.

5.4 Cubic Surfaces as Blow-ups of $\mathbb{P}^2$

A profound result in birational geometry states that every smooth cubic surface S is isomorphic to the projective plane $\mathbb{P}^2$ blown up at exactly **6 points** in general position.

Let $p_1, p_2, p_3, p_4, p_5, p_6$ be six points in $\mathbb{P}^2$. Let $\pi : S \to \mathbb{P}^2$ be the blow-up map. Let $H = \pi^*(\text{class of a line in } \mathbb{P}^2)$. The intersection lattice is completely determined by:

$$H^2 = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}$$

The canonical class of the blown-up surface is $K_S = -3H + E_1 + \cdots + E_6$, so the anticanonical class is:

$$-K_S = 3H - E_1 - \cdots - E_6$$

Why count -1-curves?

For six points in general position, the anticanonical linear system embeds S into $\mathbb{P}^3$ as a smooth cubic surface. Thus, the hyperplane class of the cubic embedding is exactly $-K_S$.

Recall the **Adjunction Formula** for a smooth curve C of genus g on a smooth surface S:

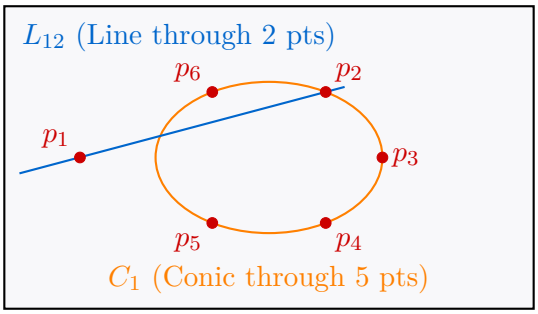
$$2g - 2 = C^2 + K_S \cdot C.$$

Since a -1-curve is isomorphic to $\mathbb{P}^1$ ($g = 0$) with $C^2 = -1$, adjunction gives $-2 = -1 + K_S \cdot C$, so $(-K_S) \cdot C = 1$. Therefore, C has degree 1 in the anticanonical embedding, meaning its geometric image is a straight line in $\mathbb{P}^3$! Conversely, any straight line $L \subset S \subset \mathbb{P}^3$ has $g = 0$ and degree $(-K_S) \cdot L = 1$, forcing $L^2 = -1$. Thus, lines on the cubic surface correspond bijectively to -1-curves on S!

As promised on Day 4, we can also compute K_S^2: each of the six blow-ups lowers K^2 by 1, so $K_S^2 = K_{\mathbb{P}^2}^2 - 6 = 9 - 6 = 3$. This matches the geometry: $(-K_S)^2 = 3$ is exactly the degree of the cubic surface, since $-K_S$ is its hyperplane class.

Let's use our intersection lattice and the self-intersection formula from Day 3 to find all the -1-curves on S!

Blowing up 6 points in $\mathbb{P}^2$



5.4.1 Counting the 27 Lines

Here is the complete inventory of the 27 lines on S:

1. The Exceptional Divisors (6 lines):

When we blow up the 6 points p_i, each point becomes an exceptional divisor E_i. By definition,

exceptional divisors are -1-curves ($E_i^2 = -1$).
Count: 6 lines.

2. Lines through 2 points (15 lines):
Take a straight line L_{ij} in $\mathbb{P}^2$ passing through exactly two points, say p_i and p_j. Its class is $H - E_i - E_j$.

$$(H - E_i - E_j)^2 = H^2 + E_i^2 + E_j^2 = 1 - 1 - 1 = -1$$

So, $\widetilde{L}_{ij}$ is a -1-curve! How many pairs of points can we choose from 6 points? $\binom{6}{2} = 15$.
Count: 15 lines.

3. Conics through 5 points (6 lines):
Take a conic section C_k in $\mathbb{P}^2$ passing through exactly five of the points (leaving out p_k). Its class is $2H - \sum_{i \neq k} E_i$.

$$\left(2H - \sum_{i \neq k} E_i\right)^2 = 4 - 5 = -1$$

So, $\widetilde{C}_k$ is also a -1-curve! How many ways can we leave out one point to select 5 points? Exactly 6.
Count: 6 lines.

Total: $6 + 15 + 6 = 27$ **lines!**

These three families produce 27 distinct -1-curves. The Cayley-Salmon theorem tells us that there are absolutely no others.

Remark 5.3 (Why No Others? A Direct Check)**.** *We can also see this without quoting Cayley–Salmon. Write a divisor class as $D = aH - \sum_{i=1}^{6} b_i E_i$. By adjunction, a -1-curve satisfies $D^2 = -1$ and $K_S \cdot D = -1$, that is,*

$$a^2 - \sum b_i^2 = -1, \qquad 3a - \sum b_i = 1.$$

By the Cauchy–Schwarz inequality, $(\sum b_i)^2 \leq 6 \sum b_i^2$, so $(3a-1)^2 \leq 6(a^2+1)$, i.e. $3a^2-6a-5 \leq 0$. Hence $a \in \{0, 1, 2\}$. Solving for the b_i in each case gives exactly the 27 classes above. (Two different irreducible curves cannot share such a class, since then $C \cdot C' = C^2 = -1 < 0$.)

5.5 The Schläfli Graph

How do these 27 lines intersect each other? We can construct a graph, known as the **Schläfli Graph**, where the 27 lines are the vertices. Two vertices are joined by an edge when the corresponding lines are **disjoint** (they do not intersect).

The intersection pattern is incredibly beautiful: each line is disjoint from exactly 16 lines, and intersects exactly 10 other lines.

Remark 5.4 (The Double-Six)**.** *A maximum collection of pairwise disjoint lines has a size of exactly 6. Such a collection is called a six. Two suitably paired sixes, say $(a_1, \ldots, a_6)$ and $(b_1, \ldots, b_6)$ where a_i intersects b_j if and only if $i \neq j$, form a classical configuration of 12 lines called a **double-six**. For example, $a_i = E_i$ and $b_j = \widetilde{C}_j$ form a double-six, since $E_i \cdot \widetilde{C}_j = 1$ for $i \neq j$ and $E_i \cdot \widetilde{C}_i = 0$.*

Problem Session: Cubic Surfaces and Intersections

1. Let's find more lines on the Fermat Cubic $x^3+y^3+z^3+w^3=0$. Using the coordinate pairings (x with y, x with z, x with w), verify how exactly 27 combinations are formed using the roots of $\omega^3=-1$.
2. On the blow-up of $\mathbb{P}^2$ at 6 points, let E_1 be the exceptional divisor at p_1, and $\widetilde{L}_{12}$ be the strict transform of the line connecting p_1 and p_2. Do E_1 and $\widetilde{L}_{12}$ intersect on the cubic surface? Use intersection theory to calculate $E_1 \cdot \widetilde{L}_{12}$.
3. *The General Position condition:* Suppose three of the six points, say p_1, p_2, p_3, happened to lie on the same straight line L in $\mathbb{P}^2$. Calculate the self-intersection of the strict transform $\widetilde{L}$ on the blown-up surface. Does the blown-up surface become singular? What happens to the cubic model?

Solutions to Problem Session

Problem 1.
The equation is $x^3 + y^3 + z^3 + w^3 = 0$. We can pair the coordinates into two groups of sums that must cancel each other out. There are exactly 3 ways to partition the variables:

- Group 1 $(x, y) \mid (z, w)$: $(x^3 + y^3) + (z^3 + w^3) = 0 \implies x = \omega_1 y, \quad z = \omega_2 w$
- Group 2 $(x, z) \mid (y, w)$: $(x^3 + z^3) + (y^3 + w^3) = 0 \implies x = \omega_1 z, \quad y = \omega_2 w$
- Group 3 $(x, w) \mid (y, z)$: $(x^3 + w^3) + (y^3 + z^3) = 0 \implies x = \omega_1 w, \quad y = \omega_2 z$

For each grouping, we must select two independent roots of -1. Since $\omega^3 = -1$ has exactly 3 complex roots, there are $3^2 = 9$ independent line choices per grouping.

$$3 \text{ groupings} \times 9 \text{ line choices} = 27 \text{ total lines!}$$

Problem 2.
Yes, they do intersect! Using the intersection lattice introduced earlier:

- E_1 is the exceptional divisor at p_1.
- $\widetilde{L}_{12}$ has the class $H - E_1 - E_2$.

We calculate their dot product using bilinearity:

$$\widetilde{L}_{12} \cdot E_1 = (H - E_1 - E_2) \cdot E_1$$

$$= (H \cdot E_1) - (E_1 \cdot E_1) - (E_2 \cdot E_1)$$

Substitute the known values ($H \cdot E_i = 0$, $E_1^2 = -1$, $E_1 \cdot E_2 = 0$):

$$= 0 - (-1) - 0 = 1$$

They intersect transversally at exactly 1 point.

Problem 3.
Let L be a line in $\mathbb{P}^2$. Its self-intersection is $H^2 = 1$. Its strict transform class is $H - E_1 - E_2 - E_3$.

$$\widetilde{L}^2 = (H - E_1 - E_2 - E_3)^2 = 1 - 1 - 1 - 1 = -2$$

The strict transform $\widetilde{L}$ becomes a -2**-curve**!

Does the blown-up surface become singular? No. The blown-up surface itself remains perfectly smooth. The issue is with the embedding into $\mathbb{P}^3$.

Using the anticanonical class $-K_S = 3H - E_1 - \cdots - E_6$ and the intersection lattice ($H^2 = 1$, $H \cdot E_i = 0$, $E_i^2 = -1$), we compute:

$$(-K_S) \cdot \widetilde{L} = (3H - E_1 - \cdots - E_6) \cdot (H - E_1 - E_2 - E_3)$$

$$= 3H^2 + E_1^2 + E_2^2 + E_3^2$$

$$= 3(1) + (-1) + (-1) + (-1) = 0$$

Because $(-K_S) \cdot \widetilde{L} = 0$, the anticanonical map contracts $\widetilde{L}$ to a single point, producing an ordinary double point on the anticanonical cubic model in $\mathbb{P}^3$.

Hence, to obtain a smooth cubic surface via the anticanonical embedding, no three of the six points may be collinear (and similarly, no six points may lie on a conic, as the strict transform of the conic would also be a -2-curve). This is the definition of "general position" in this context!

Further Reading & Resources for Day 5

To deepen your understanding of today's topics, you may refer to the following resources:

- **Miles Reid**, *Undergraduate Algebraic Geometry*: Chapter 7 is a masterpiece, dedicated entirely to the 27 lines on a cubic surface and the blow-up equivalence.
- **Robin Hartshorne**, *Algebraic Geometry*: Chapter V, Section 4 contains the rigorous proofs of the 27 lines and the Schläfli graph intersections.
- **Numberphile**, *27 Lines on a Cubic Surface* (YouTube): A fantastic, highly visual, and accessible introduction to the topic. https://www.youtube.com/watch?v=2nZbgeA2uOQ

Chapter 6

The Big Picture: The Kodaira Classification

6.1 Introduction: The Periodic Table of Surfaces

Over the past five days, we have built a powerful toolkit: intersection theory, divisors, blow-ups, minimal models, and birational invariants. We saw how curves live on surfaces and how a simple equation can hide 27 straight lines.

But algebraic geometry is not just about studying individual examples; it is about **classification**. Just as chemists organize elements into the Periodic Table, algebraic geometers organize all smooth projective surfaces into families. The **Enriques-Kodaira classification** was developed through the intuitive work of the Italian school (Enriques and Castelnuovo) and later rigorously completed and reformulated in the 1960s using the methods of Kunihiko Kodaira and modern complex geometry.

6.2 The Key to the Universe: Kodaira Dimension (κ)

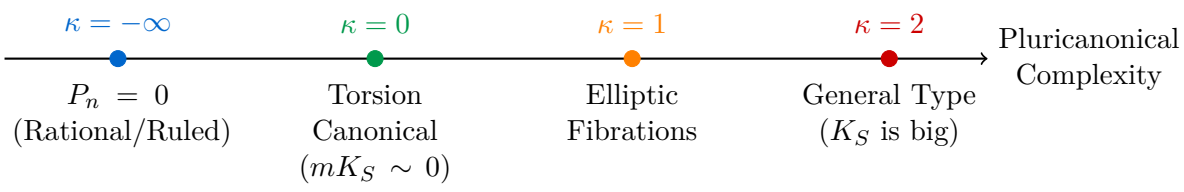
To classify smooth projective surfaces over $\mathbb{C}$, we need a single, powerful invariant that measures the "complexity" or the "asymptotic size" of the surface. We find this in the **Canonical Divisor** K_S.

Instead of just looking at K_S, we look at all its multiples nK_S for $n \geq 1$. We define the **Plurigenera** P_n as the number of independent global sections of nK_S:

$$P_n = h^0(S, \mathcal{O}_S(nK_S))$$

As n gets larger, how fast does P_n grow? The rate of this growth is called the **Kodaira Dimension**, denoted by $\kappa(S)$. For surfaces, κ can only take four possible values:

$$\kappa \in \{-\infty, 0, 1, 2\}$$



Remark 6.1 (A Note on Curvature)**.** *You may sometimes hear these categories described as having "positive," "flat," or "negative" curvature. This is only a rough geometric analogy. The Kodaira dimension directly measures the asymptotic growth of pluricanonical sections, not the pointwise sign of a metric curvature.*

6.3 The Four Realms of Surfaces

Every minimal surface falls into exactly one of these four categories.

6.3.1 The Realm of $\kappa = -\infty$ (Rational and Ruled Surfaces)

If $P_n = 0$ for all $n \geq 1$, we set $\kappa = -\infty$. This means that no positive multiple nK_S has a nonzero global section.

- **Rational Surfaces:** Birationally equivalent to $\mathbb{P}^2$. This includes $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (both minimal), and non-minimal surfaces such as smooth cubic surfaces in $\mathbb{P}^3$ (like our Fermat cubic from Day 5!), which blow down to $\mathbb{P}^2$.
- **Ruled Surfaces:** Surfaces that look like a bundle of $\mathbb{P}^1$'s over a curve of genus $g > 0$.

6.3.2 The Realm of $\kappa = 0$ (The "Torsion" Surfaces)

For a minimal surface with $\kappa = 0$, the canonical class is torsion, meaning $mK_S \sim 0$ for some integer $m \geq 1$. Therefore, P_n is bounded and periodically 0 or 1. There are exactly four types of minimal surfaces in this class:

- **K3 Surfaces:** The shining stars of modern string theory. ($p_g = 1, q = 0$, $K_S \sim 0$).
- **Abelian Surfaces:** Complex tori of dimension 2. ($p_g = 1, q = 2$, $K_S \sim 0$).
- **Enriques Surfaces:** Quotients of K3 surfaces by a fixed-point-free involution. ($p_g = 0, q = 0$, $2K_S \sim 0$ but $K_S \nsim 0$).
- **Bielliptic Surfaces:** Quotients of Abelian surfaces. ($p_g = 0, q = 1$, $mK_S \sim 0$ for $m \in \{2, 3, 4, 6\}$).

6.3.3 The Realm of $\kappa = 1$ (Elliptic Surfaces)

Here, P_n grows linearly ($P_n \sim c \cdot n$). A minimal surface with $\kappa = 1$ admits an **elliptic fibration** $f : S \to B$, whose general fiber is a smooth curve of genus 1 (an elliptic curve). Their geometry is a beautiful mix of curve theory and surface theory.

6.3.4 The Realm of $\kappa = 2$ (Surfaces of General Type)

Here, P_n grows quadratically ($P_n \sim c \cdot n^2$). These are the "heaviest" and most complicated surfaces. In many natural families of surfaces, sufficiently high-degree members are of general type. As for every surface with $\kappa \geq 0$, the minimal model of a surface of general type is unique up to isomorphism; uniqueness fails only in the realm $\kappa = -\infty$.

Summary: The Enriques-Kodaira Classification Table

Kodaira Dim.	Growth of P_n	Typical Minimal Surfaces
$\kappa = -\infty$	$P_n = 0$	Rational ($\mathbb{P}^2$) and Ruled Surfaces
$\kappa = 0$	bounded (periodically 0 or 1)	K3, Abelian, Enriques, Bielliptic
$\kappa = 1$	$P_n \sim c \cdot n$	Properly Elliptic Surfaces
$\kappa = 2$	$P_n \sim c \cdot n^2$	Surfaces of General Type

6.4 Conclusion: The End of the Beginning

Over these six days, we have journeyed from the local structure of intersection theory to the global classification of smooth projective surfaces over $\mathbb{C}$. You now possess the vocabulary and the intuition of a modern algebraic geometer. The Kodaira classification is not the end of the story—it is merely the map you will use to explore the infinite, beautiful landscapes of higher dimensions and moduli spaces.

Problem Session: The Capstone: Surfaces in $\mathbb{P}^3$

Let $S_d \subset \mathbb{P}^3$ be a smooth surface defined by a polynomial of degree d. A fundamental theorem called the *Adjunction Formula* tells us that the canonical divisor of S_d is the restriction of a line bundle from $\mathbb{P}^3$:

$$K_{S_d} = \mathcal{O}_{\mathbb{P}^3}(d-4)\big|_{S_d}$$

Recall that the global sections of $\mathcal{O}(m)$ are homogeneous polynomials of degree m. If $m < 0$, there are no polynomials of negative degree, so there are 0 sections. If $m = 0$, the only polynomials of degree 0 are constants (1 section). If $m > 0$, there are many sections.

1. Let $d = 3$ (a smooth Cubic Surface). What is its canonical class K_{S_3}? Using the plurigenera $P_n = h^0(S, nK_S)$, determine its Kodaira dimension κ. Does this match our placement of cubic surfaces in the periodic table?
2. Let $d = 4$ (a smooth Quartic Surface). What is K_{S_4}? Determine P_n for all $n \geq 1$. What is its Kodaira dimension, and which specific surface from the periodic table must it be? (Assume $q = 0$).
3. Let $d = 5$ (a smooth Quintic Surface). What is K_{S_5}? If we take nK_{S_5}, show why the number of sections P_n grows quadratically as $n \to \infty$. Determine its Kodaira dimension.

Solutions to Problem Session

Problem 1 (Degree 3: The Cubic Surface).
Using the adjunction formula for $d = 3$:

$$K_{S_3} = \mathcal{O}(3-4)|_{S_3} = \mathcal{O}(-1)|_{S_3}$$

To find the plurigenera P_n, we need the number of global sections of nK_{S_3}:

$$nK_{S_3} = \mathcal{O}(-n)|_{S_3}$$

Since $\mathcal{O}(-1)$ is anti-ample, none of its positive tensor powers has a nonzero global section.

$$P_n = h^0(S_3, \mathcal{O}(-n)) = 0 \quad \text{for all } n \geq 1.$$

Because $P_n = 0$ forever, the Kodaira dimension is defined as:

$$\kappa = -\infty$$

This perfectly matches our discussion! Smooth cubic surfaces are rational surfaces (they can be obtained by blowing up $\mathbb{P}^2$ at 6 points), and all rational surfaces reside in the $\kappa = -\infty$ realm.

Problem 2 (Degree 4: The Quartic Surface).
Using the adjunction formula for $d = 4$:

$$K_{S_4} = \mathcal{O}(4-4)|_{S_4} = \mathcal{O}(0)|_{S_4} \cong \mathcal{O}_{S_4}$$

The canonical divisor is trivial! To find the plurigenera:

$$nK_{S_4} = \mathcal{O}(0)|_{S_4}$$

The sections of $\mathcal{O}(0)$ are simply constants. There is exactly 1 independent constant (the number 1 itself). Therefore:

$$P_n = 1 \quad \text{for all } n \geq 1.$$

Because P_n is a bounded constant 1, the Kodaira dimension is:

$$\kappa = 0$$

Note also that S_4 is minimal: since $K_{S_4} \sim 0$, we have $K_{S_4} \cdot E = 0$ for every curve E, whereas a -1-curve satisfies $K \cdot E = -1$ by adjunction. So we may compare it with the minimal surfaces in our table.

Looking at our periodic table, it must be one of the four $\kappa = 0$ surfaces. We assume $q = 0$ (a fact that can be independently proven for all smooth hypersurfaces in $\mathbb{P}^3$), and since $P_1 = p_g = 1$, the fingerprints $p_g = 1, q = 0$ uniquely identify it as a **K3 Surface**! Every smooth quartic in $\mathbb{P}^3$ is a K3 surface.

Problem 3 (Degree 5: The Quintic Surface).
Using the adjunction formula for $d = 5$:

$$K_{S_5} = \mathcal{O}(5-4)|_{S_5} = \mathcal{O}(1)|_{S_5}$$

Since K_{S_5} is ample, this alone tells us the surface is of general type. It is also minimal: ampleness gives $K_{S_5} \cdot E > 0$ for every curve E, whereas a -1-curve would have $K \cdot E = -1$. To find the exact growth of the plurigenera, we look at $nK_{S_5} = \mathcal{O}(n)|_{S_5}$.

The ambient space of degree-n polynomials in 4 variables has dimension $\binom{n+3}{3}$. However, we must subtract the polynomials that vanish on our degree-5 surface S_5. Any polynomial that vanishes on S_5 is a multiple of the quintic equation, so there are $\binom{n-5+3}{3} = \binom{n-2}{3}$ such multiples.

For $n \geq 5$, after quotienting by these multiples, the dimension of the restricted sections is:

$$P_n = h^0(S_5, \mathcal{O}(n)) = \binom{n+3}{3} - \binom{n-2}{3}$$

(For $1 \leq n < 5$, the second term is interpreted as 0).

While $\binom{n+3}{3}$ grows cubically ($\sim \frac{n^3}{6}$), the difference between two cubic polynomials drops the leading term, leaving a quadratic growth:

$$P_n \sim \frac{5}{2}n^2$$

Because P_n grows at a rate proportional to n^2, the Kodaira dimension is:

$$\kappa = 2$$

This means a smooth quintic surface is a **Surface of General Type**. In fact, any smooth surface in $\mathbb{P}^3$ of degree $d \geq 5$ will have $K_S = \mathcal{O}(d-4)$ with $d - 4 > 0$, leading to $\kappa = 2$. Thus, smooth hypersurfaces in $\mathbb{P}^3$ of every degree $d \geq 5$ are of general type.

Further Reading & Resources for Day 6

To deepen your understanding of today's topics and continue your journey:

- **Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, Antonius Van de Ven**, *Compact Complex Surfaces*: The ultimate, definitive reference book for the Enriques-Kodaira Classification. Everything from K3 surfaces to general type is exhaustively detailed here.
- **Arnaud Beauville**, *Complex Algebraic Surfaces*: Chapters VI–X give a brilliant and concise proof of the classification theorem: Chapter VII introduces the Kodaira dimension, and Chapters VIII–X treat $\kappa = 0, 1, 2$.
- **Robin Hartshorne**, *Algebraic Geometry*: Chapter V is dedicated to surfaces and remains the gold standard for understanding the adjunction formula and intersection theory.