Introduction to mirror of submanifolds

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1. Outline

<u>Mirror symmetry</u>

- Strominger–Yau–Zaslow (SYZ conjecture):

  mirror symmetry of Calabi-Yau 3-folds would be explained in terms of special Lagrangian (SL) dual T^3-fibrations (including singular fibers).

$$\begin{tikzcd} X^6 \arrow[dr, "f"'] & & (X^6)^* \arrow[dl, "f^*"] \\ & B^3 & \end{tikzcd}$$

  For generic $b \in B$, $f^{-1}(b)$ and $(f^*)^{-1}(b)$ are "dual" SL T^3.

- Leung–Yau–Zaslow:

  If a SL dual torus fibration is given, "SL submanifolds" correspond to "deformed Hermitian Yang–Mills (dHYM) connections" via the **real Fourier–Mukai transform**.

In general, if X is the total space of a torus bundle, we have

$$\widetilde{\mathbf{Sub}} := \left\{ \begin{array}{c} \text{(graphical)} \\ \text{submanifolds of } X \end{array} \right\} \xrightarrow[\text{real FM}]{} \left\{ \begin{array}{c} \text{Hermitian connections} \\ \text{of } \underline{\mathbb{C}} \to X^* \end{array} \right\} =: \widetilde{\mathbf{Conn}}.$$

(X^* is given by replacing each fiber of X ($\cong T^k$) with the dual torus.)

- This correspondence is given explicitly.
- Volume of a submanifold (in the usual sense)

  $\rightsquigarrow$ "mirror" volume $\widetilde{V}$ for $\nabla \in \widetilde{\mathbf{Conn}}$.

- $\widetilde{V}$ **can be defined without torus bundle structure on** X, i.e.,

  There exists a functional V for Hermitian connections of a (general) line bundle over a (general) Riemannian manifold s.t.

$$V(\nabla) = \widetilde{V}(\nabla) \qquad \text{for} \quad \nabla \in \widetilde{\mathbf{Conn}}.$$

- Critical points of V are called **minimal connections**.
- We can show that

$$N \in \widetilde{\mathbf{Sub}} \text{ is a minimal submanifold} \qquad \Longleftrightarrow \qquad \nabla := (\text{real FM})(N) \in \widetilde{\mathbf{Conn}} \text{ is a minimal conn.}$$

In this sense, **minimal connections are "mirrors" of minimal submanifolds**.

The purpose of this course is to describe these details. The outline is as follows.

(1) Outline
(2) Review of connections
(3) The real Fourier–Mukai transform
(4) "Mirror" volume and its properties
(5) Calibrated submanifolds and their mirrors

2. Review of connections

Suppose that

- X^n: an oriented connected manifold,
- $(L, h) \longrightarrow X$: a smooth complex Hermitian line bundle.

Let me clarify the notation. First, L is a vector bundle with fiber $\mathbb{C}$ and there is a complex structure J_L, i.e., $J_L \in \Gamma(X, \mathrm{End}L)$ s.t. $J_L^2 = -\mathrm{id}_L$. J_L corresponds to the multiplication of $\sqrt{-1} \in \mathbb{C}$. Indeed, $\mathbb{C}$ acts on L by

$$\mathbb{C} \times L \longrightarrow L, \qquad (a + b\sqrt{-1}, v) \longmapsto (a + b\sqrt{-1}) \cdot v := av + bJ_L(v). \tag{2.1}$$

We can consider the case rank $L > 1$, but we only consider the case rank $L = 1$ in this course.

Also, $h \in \Gamma(X, L^* \otimes L^*)$ is the Hermitian metric of L. That is, for each $x \in X$, $h_x : L_x \times L_x \longrightarrow \mathbb{R}$ is an inner product and J_L preserves h, i.e., $h(J_L(\cdot), J_L(\cdot)) = h$.

Next, we define Hermitian connections of L. Set

$$\Omega^k(X, L) := \Gamma(X, \Lambda^k T^* X \otimes L) := \{\text{smooth sections of } \Lambda^k T^* X \otimes L\}.$$

In other words, $\Omega^k(X, L)$ is the space of L-valued k-forms. Note that $\Lambda^k T^* X \otimes L = \Lambda^k T^* X \otimes_{\mathbb{R}} L$ admits a $\mathbb{C}$-action induced from (2.1).

Definition 2.1. A map $\nabla : \Omega^0(X, L) \to \Omega^1(X, L)$ is called a **Hermitian connection** if

(1) ∇ is $\mathbb{C}$-linear, i.e.,

$$\nabla\big((a + b\sqrt{-1}) \cdot s\big) = (a + b\sqrt{-1}) \cdot \nabla s$$

for any $a, b \in \mathbb{R}$ and $s \in \Gamma(X, L)$.

(2) ∇ satisfies the Leibnitz rule, i.e.,

$$\nabla(fs) = df \otimes s + f \nabla s$$

for any smooth function $f \in \Omega^0(X)$ and $s \in \Gamma(X, L)$.

(3) $\nabla h = 0$, i.e.,

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

for any $s_1, s_2 \in \Gamma(X, L)$.

Set

$$\mathcal{A}_0 := \{\text{Hermitian connections of } (L,h)\}.$$

Lemma 2.2. *For any fixed $\nabla_0 \in \mathcal{A}_0$, we have*

$$\mathcal{A}_0 = \nabla_0 + \Omega^1(X, \mathrm{End}_{\textit{skew-Herm}}(L)) = \nabla_0 + \sqrt{-1}\Omega^1(X)\cdot \mathrm{id}_L,$$

Note that

$$\begin{aligned}\Omega^1(X, \mathrm{End}_{\textit{skew-Herm}}(L)) =&\Gamma(X, T^*X \otimes \mathrm{End}_{\textit{skew-Herm}}(L)),\\ \mathrm{End}_{\textit{skew-Herm}}(L) :=&\{T : L \longrightarrow L \mid T \textit{ is } \mathbb{C}\textit{-linear},\ h(T(\cdot),\cdot) + h(\cdot, T(\cdot)) = 0\} \overset{(*)}{=} \sqrt{-1}\mathbb{R}\cdot \mathrm{id}_L.\end{aligned}$$

Proof. First we show (*). Recall that $\mathrm{End}(L) := \{T : L \longrightarrow L \mid T \text{ is } \mathbb{C}\text{-linear}\}$ has a global section id_L. Since L is a line bundle, any element T of $\mathrm{End}(L)$ is of the form $T = z \cdot \mathrm{id}_L$ for $z = x + y\sqrt{-1} \in \mathbb{C}$. Then for $u, v \in L$, we have

$$\begin{aligned}h(T(u), v) &= h(xu + yJ_L(u), v),\\ h(u, T(v)) &= h(u, xv + yJ_L(v)) = xh(u, v) - yh(J_L(u), v).\end{aligned}$$

Thus

$$h(T(u), v) + h(u, T(v)) = 0 \qquad \Longleftrightarrow \qquad 2xh(u, v) = 0.$$

Since u, v is arbitrary, we see that $x = 0$ and obtain $\mathrm{End}_{\text{skew-Herm}}(L) \subset \sqrt{-1}\mathbb{R}\cdot \mathrm{id}_L$. The converse is easy to show.

Take any $\nabla_1 \in \mathcal{A}_0$. By (2) of Definition 2.1, we see that

$$(\nabla_1 - \nabla_0)(fs) = f(\nabla_1 - \nabla_0)s.$$

This means that $\nabla_1 - \nabla_0$ is a tensor, i.e., there is $T \in \Omega^1(X, \mathrm{End}(L))$ s.t. $\nabla_1 = \nabla_0 + T$.

By (3), T satisfies $h(Ts_1, s_2) + h(s_1, Ts_2) = 0$, which implies that $T \in \Omega^1(X, \mathrm{End}_{\text{skew-Herm}}(L))$. □

From $\nabla \in \mathcal{A}_0$, we can define the **exterior covariant derivative** $d^\nabla : \Omega^p(X, L) \longrightarrow \Omega^{p+1}(X, L)$ by

$$\begin{aligned}d^\nabla s &= \nabla s && \text{for} \quad s \in \Omega^0(X, L),\\ d^\nabla(\alpha \wedge s) &= d\alpha \wedge s + (-1)^k \alpha \wedge d^\nabla s && \text{for} \quad \alpha \in \Omega^k(X) \quad \text{and} \quad s \in \Omega^\ell(X, L).\end{aligned}$$

When $\ell = 0$, we consider $\alpha \wedge s = a \otimes s$.

Lemma 2.3. *For a smooth function $f \in \Omega^0(X)$ and $s \in \Gamma(X, L)$, we have*

$$(d^\nabla \circ d^\nabla)(fs) = f(d^\nabla \circ d^\nabla)(s) \in \Omega^2(X, L).$$

Thus $d^\nabla \circ d^\nabla$ is a tensor. We call

$$F_\nabla := d^\nabla \circ d^\nabla \in \Omega^2(X, \mathrm{End}(L))$$

the **curvature** of $\nabla \in \mathcal{A}_0$.

Proof. We compute

$$\begin{aligned}(d^\nabla \circ d^\nabla)(fs) =& d^\nabla(df \otimes s + f d^\nabla s)\\ =& -df \wedge d^\nabla s + df \wedge d^\nabla s + f(d^\nabla \circ d^\nabla)(s) = f(d^\nabla \circ d^\nabla)(s).\end{aligned}$$

□

For $\xi \in \Omega^k(X, L)$ and $\eta \in \Omega^\ell(X, L)$, define

$$h(\xi, \eta) \in \Omega^{k+\ell}(X)$$

by taking the metric for L-parts and the wedge product for differential form parts. That is, setting $\xi = \sum_i \alpha_i \otimes s_i$ and $\eta = \sum_j \alpha'_j \otimes s'_j$ for $\alpha_i \in \Omega^k(X)$, $\alpha'_j \in \Omega^\ell(X)$ and $s_i, s'_j \in \Gamma(X, L)$, we have

$$h(\xi, \eta) = \sum_{i,j} \alpha_i \wedge \alpha'_j h(s_i, s'_j).$$

Then we can show the following.

Fact 2.4. *For $\xi \in \Omega^k(X, L)$ and $\eta \in \Omega^\ell(X, L)$, we have*

$$dh(\xi, \eta) = h(d^\nabla \xi, \eta) + (-1)^k h(\xi, d^\nabla \eta).$$

This is proved by a straightforward computation and we omit the proof. Using this fact, we can see the following.

Lemma 2.5. *We have*

$$F_\nabla \in \Omega^2(X, \mathrm{End}_{\textit{skew-Herm}}(L)) = \sqrt{-1}\Omega^2(X) \cdot \mathrm{id}_L,$$

and we may set

$$F_\nabla = \sqrt{-1} E_\nabla \cdot \mathrm{id}_L \qquad \textit{for} \quad E_\nabla \in \Omega^2(X).$$

Proof. Recall that

$$dh(s_1, s_2) = h(\underbrace{\nabla s_1}_{=d^\nabla s_1}, s_2) + h(s_1, \underbrace{\nabla s_2}_{=d^\nabla s_2})$$

for any $s_1, s_2 \in \Gamma(X, L)$. Taking d on both sides, we obtain

$$\begin{aligned}0 =& h(d^\nabla d^\nabla s_1, s_2) - h(d^\nabla s_1, d^\nabla s_2) + h(d^\nabla s_1, d^\nabla s_2) + h(s_1, d^\nabla d^\nabla s_2)\\ =& h(F_\nabla(s_1), s_2) + h(s_1, F_\nabla(s_2)),\end{aligned}$$

which implies that $F_\nabla \in \Omega^2(X, \mathrm{End}_{\text{skew-Herm}}(L))$. □

We also see the following.

Lemma 2.6. *For any $\xi \in \Omega^k(X,L)$, we have*

$$(d^\nabla \circ d^\nabla)(\xi) = F_\nabla \wedge \xi.$$

This notation means that we take the wedge product for differential form parts of F_∇ and ξ, and also take the composition of the $\mathrm{End}_{\text{skew-Herm}}(L)$ part of F_∇ and the L part of ξ.

Proof. We only have to show this for $\xi = \alpha \otimes s$ for $\alpha \in \Omega^k(X)$ and $s \in \Gamma(X,L)$ because ξ is written as a finite sum of these locally. We compute

$$\begin{aligned}(d^\nabla \circ d^\nabla)(\xi) =& d^\nabla(d\alpha \otimes s + (-1)^k \alpha \wedge d^\nabla s)\\ =&(-1)^{k+1} d\alpha \wedge d^\nabla s + (-1)^k d\alpha \wedge d^\nabla s + \alpha \wedge (d^\nabla \circ d^\nabla)(s)\\ =&\alpha \wedge \underbrace{(d^\nabla \circ d^\nabla)}_{=F_\nabla}(s) = F_\nabla \wedge (\alpha \wedge s).\end{aligned}$$

□

Proposition 2.7 (Bianchi identity)**.** *We have $dE_\nabla = 0$ for any $\nabla \in \mathcal{A}_0$.*

Proof. For $s \in \Gamma(X,L)$, we compute

$$\begin{aligned}(d^\nabla \circ d^\nabla \circ d^\nabla)(s) =& d^\nabla\big((d^\nabla \circ d^\nabla)(s)\big)\\ =& d^\nabla(F_\nabla \otimes s) = d^\nabla(\sqrt{-1}E_\nabla \otimes s) = \sqrt{-1} dE_\nabla \otimes s + \sqrt{-1}E_\nabla \wedge d^\nabla s,\\ (d^\nabla \circ d^\nabla \circ d^\nabla)(s) =& (d^\nabla \circ d^\nabla)(d^\nabla s) = F_\nabla \wedge d^\nabla s = \sqrt{-1}E_\nabla \wedge d^\nabla s.\end{aligned}$$

Hence we have $dE_\nabla \otimes s = 0$. Since s is arbitrary, we see that $dE_\nabla = 0$. □

We also see the following.

Lemma 2.8. *For any $\nabla \in \mathcal{A}_0$ and $a \in \Omega^1(X)$, we have*

$$F_{\nabla + \sqrt{-1}a \cdot \mathrm{id}_L} = F_\nabla + \sqrt{-1} da \cdot \mathrm{id}_L.$$

Proof. Set $\nabla' = \nabla + \sqrt{-1}a \cdot \mathrm{id}_L$. For any $s \in \Gamma(X,L)$, we compute

$$d^{\nabla'} s = \nabla' s = \underbrace{\nabla s}_{=d^\nabla s} + \sqrt{-1} a \otimes s.$$

In addition, for any $\xi \in \Omega^k(X,L)$, we see that

$$d^{\nabla'}\xi = d^\nabla \xi + \sqrt{-1} a \wedge \xi \tag{2.2}$$

Indeed, we only have to show this for $\xi = \alpha \otimes s$ for $\alpha \in \Omega^k(X)$ and $s \in \Gamma(X,L)$ as above. We compute

$$d^{\nabla'}\xi = d\alpha \otimes s + (-1)^k \alpha \wedge \underbrace{\nabla' s}_{=d^\nabla s + \sqrt{-1} a \otimes s} = d^\nabla \xi + \sqrt{-1} a \wedge \underbrace{\alpha \otimes s}_{=\xi}.$$

Hence we obtain (2.2). Thus

$$\begin{aligned} d^{\nabla'}(d^{\nabla'}s) =& d^{\nabla'}(d^{\nabla}s+\sqrt{-1}a\otimes s) \\ =& \underbrace{d^{\nabla'}(d^{\nabla}s)}_{=d^{\nabla}(d^{\nabla}s)+\sqrt{-1}a\wedge d^{\nabla}s} +\sqrt{-1}da\otimes s-\sqrt{-1}a\wedge \underbrace{d^{\nabla'}s}_{=d^{\nabla}s+\sqrt{-1}a\otimes s} = F_{\nabla}\otimes s+\sqrt{-1}da\otimes s. \end{aligned}$$

□

Next, we study flat connections.

Definition 2.9. A Hermitian connection $\nabla\in\mathcal{A}_0$ is called **flat** if $E_{\nabla}=0$.

A Hermitian line bundle (L,h) is called **flat** if it admits a flat connection.

Example 2.10. A trivial bundle $L=X\times\mathbb{C}$ with the product metric, where $\mathbb{C}$ is endowed with the standard flat metric, is a flat line bundle.

Indeed, the exterior derivative d defines a flat connection. That is, since the section of L is a $\mathbb{C}$-valued function, we can define a connection of L by

$$\nabla s = ds \qquad \text{for} \quad s\in\Omega^0(X,L)=\Omega^0(X,\mathbb{C}):=\{X\longrightarrow\mathbb{C}\text{: a smooth map}\}.$$

Then this ∇ is a Hermitian connection and satisfies $d^{\nabla}\circ d^{\nabla}=d\circ d=0$.

Lemma 2.11. *Let ∇_0 be a flat connection of a flat line bundle L. Then any flat Hermitian connection of L is of the form $\nabla_0+\sqrt{-1}a\cdot\mathrm{id}_L$ for a* **closed** *1-form $a\in\Omega^1(X)$.*

Proof. Recall that any element of $\mathcal{A}_0$ is of the form $\nabla_0+\sqrt{-1}a\cdot\mathrm{id}_L$ for $a\in\Omega^1(X)$. By Lemma 2.8, we have

$$F_{\nabla_0+\sqrt{-1}a\cdot\mathrm{id}_L}=F_{\nabla_0}+\sqrt{-1}da\cdot\mathrm{id}_L=\sqrt{-1}da\cdot\mathrm{id}_L,$$

which implies that $da=0$. □

There is a canonical group acting on $\mathcal{A}_0$. Let $\mathcal{G}_U$ be the group of unitary gauge transformations of (L,h). Precisely,

$$\mathcal{G}_U=\{\,f\cdot\mathrm{id}_L \mid f\in\Omega^0(X,\mathbb{C}),\ |f|=1\,\}\cong C^{\infty}(X,S^1).$$

The action $\mathcal{G}_U\times\mathcal{A}_0\to\mathcal{A}_0$ is defined by

$$(\lambda,\nabla)\longmapsto\lambda^{-1}\circ\nabla\circ\lambda.$$

More explicitly, for $\lambda=f\cdot\mathrm{id}_L$ and $s\in\Gamma(X,L)$, we have

$$(\lambda^{-1}\circ\nabla\circ\lambda)(s)=f^{-1}\nabla(fs)=f^{-1}(df\otimes s+f\nabla s)=\nabla s+f^{-1}df\otimes s,$$

and hence,

$$\lambda^{-1}\circ\nabla\circ\lambda=\nabla+f^{-1}df\otimes\mathrm{id}_L.$$

Thus, the $\mathcal{G}_U$-orbit through $\nabla \in \mathcal{A}_0$ is given by $\nabla + \mathcal{K}_U \cdot \mathrm{id}_L$, where

$$\mathcal{K}_U := \{\, f^{-1}df \in \sqrt{-1}\Omega^1 \mid f \in \Omega^0(X, \mathbb{C}),\ |f| = 1 \,\}.$$

Lemma 2.12. *For any $\nabla \in \mathcal{A}_0$, the curvature 2-form F_∇ is invariant under the action of $\mathcal{G}_U$.*

Proof. This statement says that $F_{\nabla + f^{-1}df \otimes \mathrm{id}_L} = F_\nabla$ for any $f \in \Omega^0(X, \mathbb{C})$ with $|f| = 1$. By Lemma 2.8, we see that

$$F_{\nabla + f^{-1}df \otimes \mathrm{id}_L} = F_\nabla + d(f^{-1}df) \otimes \mathrm{id}_L = F_\nabla.$$

□

Denote by $\mathcal{A}_{\mathrm{flat}}$ the space of flat Hermitian connections:

$$\mathcal{A}_{\mathrm{flat}} = \{\nabla \in \mathcal{A}_0 \mid F_\nabla = 0\}.$$

Lemma 2.12 implies that $\mathcal{G}_U$ acts on $\mathcal{A}_{\mathrm{flat}}$. Then we have the following.

Fact 2.13. *Let L be a flat line bundle with a flat connection ∇_0. Then*

$$\mathcal{A}_{\mathit{flat}}/\mathcal{G}_U \xrightarrow{\cong} H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z}), \qquad [\nabla_0 + \sqrt{-1}a \cdot \mathrm{id}_L] \longmapsto [a],$$

where we identify $H^1(X, \mathbb{Z})$ with its image in $H^1_{dR}(X) = H^1(X, \mathbb{R})$, that is,

$$H^1(X, \mathbb{Z}) = \left\{ [\alpha] \in H^1_{dR}(X) \;\middle|\; \int_A \alpha \in \mathbb{Z} \text{ for any } A \in H_1(X, \mathbb{Z}) \right\}.$$

By Lemmas 2.11 and 2.12, we see that this is well-defined and surjective.

The injectivity is a little bit complicated. For example, this follows from Lemma 4.1 of

- K. Kawai and H. Yamamoto, Mirror of volume functionals on manifolds with special holonomy. Adv. Math. 405 (2022), Paper No. 108515, 69 pp.

3. The real Fourier–Mukai transform

In this section, we introduce the **real Fourier–Mukai transform** for a torus fibration, which gives the "mirror" correspondence, and give some computations using it.

For simplicity, we consider the following case:

$$X = B^k \times T^n, \qquad X^* = B^k \times (T^n)^*,$$

where $B^k \subset \mathbb{R}^k$ is an open set and

$$T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n, \qquad (T^n)^* = (\mathbb{R}^n)^*/2\pi(\mathbb{Z}^n)^*$$

and $(\mathbb{Z}^n)^* = \{\alpha \in (\mathbb{R}^n)^* \mid \langle \alpha, v \rangle \in \mathbb{Z} \text{ for } \forall v \in \mathbb{Z}^n\}$.

The idea is:

(1) First, we assign $a \in T^n$ to a Hermitian connection ∇^a of $(T^n)^* \times \mathbb{C} \to (T^n)^*$.

(2) Using this, we have

$$\left\{\begin{array}{c}\text{graphical}\\ \text{submanifolds of } X\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{maps}\\ B^k \to T^n\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Hermitian connctions}\\ \text{of } X^* \times \mathbb{C} \to X^*\end{array}\right\}$$

$$\operatorname{graph}(f) \longleftrightarrow f \longmapsto \nabla := \left\{\nabla^{f(x)}\right\}_{x \in B^k}$$

where $\operatorname{graph}(f) := \{(x, f(x)) \in X \mid x \in B^k\}$.

This is the real Fourier–Mukai transform. In this sense, **the real Fourier–Mukai transform gives the correspondence between graphical submanifolds of X and the Hermitian connections of $X^* \times \mathbb{C} \to X^*$.**

The correspondence (1) is given by the following identification:

$$\begin{aligned} T^n &= \mathbb{R}^n / 2\pi\mathbb{Z}^n \\ &\cong H^1((T^n)^*, \mathbb{R}) / 2\pi H^1((T^n)^*, \mathbb{Z}) \\ &\cong \{\text{flat Hermitian connections of } (T^n)^* \times \mathbb{C} \to (T^n)^*\} / \mathcal{G}_U. \end{aligned}$$

Explicitly,

$$(a^1, \cdots, a^n) \longmapsto \left[\sum_{j=1}^n a^j dy^j\right] \longmapsto \left[d + \sqrt{-1}\sum_{j=1}^n a^j dy^j\right],$$

where $(y^1, \cdots, y^n)$ are coordinates on $(T^n)^*$.

(2) Given a map $f = (f^1, \cdots, f^n) : B^k \longrightarrow T^n$, we obtain a Hermitian connection $\nabla := \left\{\nabla^{f(x)}\right\}_{x \in B^k}$:

$$\nabla = d + \sqrt{-1}\sum_{j=1}^n f^j(x) dy^j.$$

We call ∇ the **real Fourier–Mukai transform** of f (or $\operatorname{graph}(f)$).

Note that ∇ is defined up to $\mathcal{G}_U$-action, or in other words, up to the addition of elements of $2\pi\mathbb{Z}^n$. That is, we may replace $(f^1, \cdots, f^n)$ with $(f^1 + z^1, \cdots, f^n + z^n)$ for $(z^1, \cdots, z^n) \in 2\pi\mathbb{Z}^n$. But the curvature

$$F_\nabla = \sqrt{-1}\sum_{j=1}^n df^j \wedge dy^j$$

is independent of this addition.

By this explicit correspondence, we might be able to import many notions for submanifolds to the connection side. Let's do some explicit calculations here.

First, we will describe the volume form of $\mathrm{graph}(f)$ in terms of the real Fourier–Mukai transform ∇. Let us introduce the notation.

- $(x^1, \cdots, x^k)$: coordinates of $B^k \subset \mathbb{R}^k$,
- $f = (f^{k+1}, \cdots, f^{k+n}) : B^k \longrightarrow T^n$ (We change the index slightly.),
- $(y^{k+1}, \cdots, y^{k+n})$: coordinates of T^n.

The graph of f is given by

$$\mathrm{graph}(f) := \{\, (x, f(x)) \mid x \in B \,\},$$

which is a k-dimensional submanifold of $X = B^k \times T^n$.

The real Fourier–Mukai transform ∇ of f is given by

$$\nabla = d + \sqrt{-1} \sum_{a=k+1}^{k+n} f^a dy^a, \qquad F_\nabla = \sqrt{-1} E_\nabla = \sqrt{-1} \sum_{i=1}^{k} \sum_{a=k+1}^{k+n} \frac{\partial f^a}{\partial x^i} dx^i \wedge dy^a.$$

Define

$$\iota : B^k \longrightarrow X = B^k \times T^n, \qquad \iota(x) = (x, f(x)).$$

Set

$$\partial_i := \frac{\partial}{\partial x^i}, \qquad \partial_a := \frac{\partial}{\partial y^a}, \qquad f_i^a := \frac{\partial f^a}{\partial x^i}$$

for $1 \leq i \leq k$ and $k+1 \leq a \leq k+n$, and

$$v_i := \iota_*(\partial_i) = \partial_i + \frac{\partial f}{\partial x^i} = \partial_i + \sum_{a=k+1}^{k+n} f_i^a \partial_a \qquad \text{for} \quad i = 1, \cdots, k.$$

Denote by $\langle \cdot, \cdot \rangle$ the standard metric on $X = B^k \times T^n$, i.e.,

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^{k} dx^i \otimes dx^i + \sum_{a=k+1}^{k+n} dy^a \otimes dy^a.$$

Recall that the volume form on B^k w.r.t. the induced metric $\iota^* \langle \cdot, \cdot \rangle$ is given by

$$\sqrt{\det(\langle v_i, v_j \rangle)_{i,j=1,\cdots,k}} \cdot dx^1 \wedge \cdots \wedge dx^k.$$

Proposition 3.1. *Define* $E_\nabla^\sharp \in \Gamma(X, \mathrm{End}_{skew}(TX))$ *by*

$$\langle E_\nabla^\sharp(\cdot), \cdot \rangle = E_\nabla, \qquad i.e., \qquad E_\nabla^\sharp = \sum_{i=1}^{k} \sum_{a=k+1}^{k+n} f_i^a \big(dx^i \otimes \partial_a - dy^a \otimes \partial_i \big).$$

Then we have

$$\sqrt{\det(\langle v_i, v_j \rangle)} = \sqrt{\det\Big(\mathrm{id}_{TX} + E_\nabla^\sharp \Big)}.$$

We observe that the right hand side is also defined for any Hermitian connection on a (general) Hermitian line bundle (which does not admit a torus bundle structure).

Proof. Since $v_i = \iota_*(\partial_i) = \partial_i + \frac{\partial f}{\partial x^i} = \partial_i + \sum_{a=k+1}^{k+n} f_i^a \partial_a$, we have

$$\langle v_i, v_j \rangle = \delta_{ij} + \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle = \delta_{ij} + \sum_{a=k+1}^{k+n} f_i^a f_j^a.$$

Hence

$$\det(\langle v_i, v_j \rangle) = \det\bigl(I_k + {}^tAA\bigr),$$

where I_k is the identity matrix of dimension k, A is a $n \times k$ matrix defined by

$$A = (f_i^a)_{k+1 \le a \le k+n, 1 \le i \le k}$$

and ${}^tA$ is the transpose of A.

Now fix $x \in B$ and consider the value at x. Since ${}^tAA$ is symmetric, it is diagonalizable with real eigenvalues $\{\lambda_1, \cdots, \lambda_k\}$. Note that $\lambda_j \ge 0$ for each j because for any $v \in \mathbb{R}^k$,

$$\langle {}^tAAv, v \rangle = |Av|^2 \ge 0.$$

We also have

$$E_\nabla^\sharp = \begin{pmatrix} 0 & -{}^tA \\ A & 0 \end{pmatrix}, \qquad (E_\nabla^\sharp)^2 = \begin{pmatrix} -{}^tAA & 0 \\ 0 & -A{}^tA \end{pmatrix},$$
$${}^t\Bigl(\mathrm{id}_{TX} + E_\nabla^\sharp\Bigr)\Bigl(\mathrm{id}_{TX} + E_\nabla^\sharp\Bigr) = \mathrm{id}_{TX} - (E_\nabla^\sharp)^2.$$

Thus for the computation, we should know eigenvalues of $A{}^tA$.

Lemma 3.2. *We have*

$$\{\textit{nonzero eigenvalues of } {}^tAA\} = \{\textit{nonzero eigenvalues of } A{}^tA\}.$$

Proof. Let $\lambda \neq 0$ be an eingenvalue of ${}^tAA$. Take an eigenvector $0 \neq v \in \mathbb{R}^k$, i.e., ${}^tAAv = \lambda v$. Then

$$A{}^tA(Av) = A({}^tAAv) = \lambda Av.$$

If $Av = 0$, we have $0 = {}^tAAv = \lambda v$. This is impossible since $\lambda, v \neq 0$. Thus $Av \neq 0$ and λ is an eigenvalue of $A{}^tA$.

The reverse inclusion also holds by replacing A with ${}^tA$. □

Thus assuming $\lambda_1, \cdots, \lambda_\ell \neq 0$, $\lambda_{\ell+1} = \cdots = \lambda_k = 0$, we see that

$$\det\bigl(I_k + {}^tAA\bigr) = (1 + \lambda_1) \cdots (1 + \lambda_\ell).$$

We also compute

$$\det\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)^2=\det\Big({}^t\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)\Big)$$
$$=\det\begin{pmatrix} I+{}^tAA & 0 \\ 0 & I+A^tA\end{pmatrix}=(1+\lambda_1)^2\cdots(1+\lambda_\ell)^2.$$

Note that $\det\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)>0$ because $E_\nabla^\sharp$ is skew-symmetric so it is conjugate to

$$\begin{pmatrix} 0 & -\mu_1 \\ \mu_1 & 0\end{pmatrix}\oplus\begin{pmatrix} 0 & -\mu_2 \\ \mu_2 & 0\end{pmatrix}\oplus\cdots$$

for $\mu_j\in\mathbb{R}$, so $\det\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)=(1+\mu_1^2)(1+\mu_2^2)\cdots>0$. Hence the proof is completed. $\square$

We give another computation. When $S:=\mathrm{graph}(f)$ is a minimal submanifold, we will see what condition is imposed for its real Fourier–Mukai transform ∇.

Recall that $E_\nabla^\sharp=\sum_{i=1}^k\sum_{a=k+1}^{k+n}f_i^a(dx^i\otimes\partial_a-dy^a\otimes\partial_i)$. Then observe that

$$v_i=\partial_i+\sum_{a=k+1}^{k+n}f_i^a\partial_a=(\mathrm{id}_{TX}+E_\nabla^\sharp)(\partial_i)$$

for $1\le i\le k$. Set

$$\eta_a:=(\mathrm{id}_{TX}+E_\nabla^\sharp)(\partial_a)=\partial_a-\sum_{j=1}^k f_j^a\partial_j$$

for $k+1\le a\le k+n$.

Lemma 3.3. *$\{v_i\}_{i=1}^k$ spans TS and $\{\eta_a\}_{a=k+1}^{k+n}$ spans the orthogonal complement (normal bundle) $T^\perp S$.*

Proof. It is clear that $\{v_i\}_{i=1}^k$ spans TS by the definition of $S=\mathrm{graph}(f)$. We can compute

$$\langle v_i,\eta_a\rangle=\left\langle \partial_i+\sum_{b=k+1}^{k+n}f_i^b\partial_b\ ,\ \partial_a-\sum_{j=1}^k f_j^a\partial_j\right\rangle=-f_i^a+f_i^a=0$$

for any $i=1,\cdots,k$ and $a=k+1,\cdots,k+n$. $\square$

We denote the induced metric by $g=(g_{ij})$:

$$g_{ij}=\langle v_i,v_j\rangle=\delta_{ij}+f_i^af_j^a\qquad\text{and set}\qquad g^{-1}=(g^{ij}).$$

Also, set

$$G_\nabla:={}^t\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)\Big(\mathrm{id}_{TX}+E_\nabla^\sharp\Big)=\mathrm{id}_{TX}-(E_\nabla^\sharp)^2.$$

Define a differential operator $\delta_\nabla : \Omega^p(X) \longrightarrow \Omega^{p-1}(X)$ by

$$\delta_\nabla \alpha := -\sum_{i=1}^{k} i(G_\nabla^{-1}(\partial_i))D_{\partial_i}\alpha - \sum_{a=k+1}^{k+n} i(G_\nabla^{-1}(\partial_a))D_{\partial_a}\alpha,$$

where D is the (flat) Levi-Civita connection of $\langle \cdot, \cdot \rangle$.

Proposition 3.4. *The graph f is minimal if and only if $\delta_\nabla E_\nabla = 0$.*

We observe that $\delta_\nabla E_\nabla$ is defined for any Hermitian connection on a (general) Hermitian line bundle (which does not admit a torus bundle structure).

Proof. Recall

$$\iota : B^k \longrightarrow X = B^k \times T^n, \qquad \iota(x) = (x, f(x)).$$

Then the graph f is minimal if and only if the mean curvature

$$H := \sum_{i,j=1}^{k} g^{ij} \left(D_{\partial_i}^{\iota^* TX}(\iota_*(\partial_j)) \right)^{\perp}$$

vanishes, where $D^{\iota^* TX}$ is the induced connection on the pullback $\iota^* TX$ from the Levi-Civita connection D of $\langle \cdot, \cdot \rangle$ by ι, and $\perp$: $\iota^* TX = TS \oplus T^\perp S \to T^\perp S$ is the orthogonal projection. Since

$$D_{\partial_i}^{\iota^* TX}(\iota_*(\partial_j)) = D_{\partial_i}^{\iota^* TX}\left(\left(\partial_j + \sum_{a=k+1}^{k+n} f_j^a \partial_a \right) \circ \iota \right) = \sum_{a=k+1}^{k+n} f_{ij}^a \partial_a \circ \iota, \qquad \text{where} \qquad f_{ij}^a := \frac{\partial^2 f^a}{\partial x^i \partial x^j},$$

by $D_{\partial_i}^{\iota^* TX}(\iota_*(\partial_j)) = (D_{\partial_i}\partial_j) \circ \iota = 0$, etc., we have

$$\left\langle D_{\partial_i}^{\iota^* TX}(\iota_*(\partial_j)), \eta_a \right\rangle = f_{ij}^a$$

for $k+1 \leq a \leq k+n$. Since $\{\eta_a\}_{a=k+1}^{k+n}$ spans $T^\perp S$, it follows that $H = 0$ if and only if

$$\sum_{i,j=1}^{k} g^{ij} f_{ij}^a = 0 \qquad \text{for any } k+1 \leq a \leq k+n. \tag{3.1}$$

Next, we compute $\delta_\nabla E_\nabla$. Since

$$E_\nabla^\sharp = \sum_{i=1}^{k} \sum_{a=k+1}^{k+n} f_i^a (dx^i \otimes \partial_a - dy^a \otimes \partial_i),$$

we have

$$G_\nabla = \mathrm{id}_{TX} - (E_\nabla^\sharp)^2 = \sum_{i,j=1}^{k} \left(\delta_{ij} + \sum_{a=k+1}^{k+n} f_i^a f_j^a \right) dx^i \otimes \partial_j + \sum_{a,b=k+1}^{k+n} \left(\delta_{ab} + \sum_{i=1}^{k} f_i^a f_i^b \right) dy^a \otimes \partial_b.$$

Then it follows that G_∇^{-1} is a linear combination of $dx^i \otimes \partial_j$ and $dy^a \otimes \partial_b$. Using this, we compute

$$\begin{aligned}
&\delta_\nabla E_\nabla \\
&= -\sum_{i=1}^{k} i(G_\nabla^{-1}(\partial_i)) D_{\partial_i}\left(\sum_{j=1}^{k}\sum_{a=k+1}^{k+n} f_j^a dx^j \wedge dy^a\right) - \sum_{a=k+1}^{k+n} i(G_\nabla^{-1}(\partial_a))\, \underbrace{D_{\partial_a}\left(\sum_{j=1}^{k}\sum_{a=k+1}^{k+n} f_j^a dx^j \wedge dy^a\right)}_{=0} \\
&= -\sum_{i,j=1}^{k}\sum_{a=k+1}^{k+n} f_{ij}^a dx^j(G_\nabla^{-1}(\partial_i)) dy^a,
\end{aligned}$$

where we use the fact that f^a is a function of $(x^1, \cdots, x^k)$.

Since $g_{ij} = \langle v_i, v_j\rangle = \langle G_\nabla(\partial_i), \partial_j\rangle$, we have

$$dx^j(G_\nabla^{-1}(\partial_i)) = \langle G_\nabla^{-1}(\partial_i), \partial_j\rangle = g^{ij},$$

and hence

$$\delta_\nabla E_\nabla = -\sum_{i,j=1}^{k}\sum_{a=k+1}^{k+n} g^{ij} f_{ij}^a dy^a. \tag{3.2}$$

Then by (3.1) and (3.2), the proof is completed. □

4. "Mirror" volume and its properties

Until now we have considered the case when the manifold is $B^k \times T^n$. The calculations at the end of the previous section imply that the "volume functional" can be defined in a more general situation. Indeed, we can define as follows.

Suppose that

- (X^n, g): a compact oriented connected Riemannian manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle,
- $\mathcal{A}_0 = \{$Hermitian connections of $(L, h)\}$.

For each $\nabla \in \mathcal{A}_0$, define $E_\nabla^\sharp \in \Gamma(X, \mathrm{End}_{\mathrm{skew}}(TX))$ by

$$g(E_\nabla^\sharp(\cdot), \cdot) = E_\nabla.$$

Definition 4.1. Define the **volume functional** $V : \mathcal{A}_0 \longrightarrow \mathbb{R}$ by

$$V(\nabla) := \int_X v(\nabla)\mathrm{vol}_g, \qquad v(\nabla) := \sqrt{\det\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)}.$$

The description of $v(\nabla)$ is given in Proposition 3.1, where we describe the volume form of the graphical submanifold in terms of its real Fourier–Mukai transform. So in this sense, V can be considered as the "mirror" of the (standard) volume functional for submanifolds.

V is called the **Dirac-Born-Infeld (DBI) action** in physics.

As before, we can define the following tensor. (Recall that ${}^tE_\nabla^\sharp = -E_\nabla^\sharp$.)

$$G_\nabla := {}^t\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right) \circ \left(\mathrm{id}_{TX} + E_\nabla^\sharp\right) = \mathrm{id}_{TX} - E_\nabla^\sharp \circ E_\nabla^\sharp.$$

The tensor G_∇ is useful for our computation.

Lemma 4.2. *(1)*

$$v(\nabla) = \sqrt{\det\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)} = \sqrt{1 + |E_\nabla|^2 + \left|\frac{E_\nabla^2}{2!}\right|^2 + \left|\frac{E_\nabla^3}{3!}\right|^2 + \cdots}.$$

(2) $v(\nabla) \geq 1$.

(3) G_∇ *is positive definite.*

($\Longrightarrow$ G_∇ *is considered as the "metric" deformed by* ∇*.)*

(4) $v(\nabla) = (\det G_\nabla)^{1/4}$.

Proof. (1) Fix $x \in X$ and consider the value at x. For simplicity, we assume that $\dim X = 6$. Since $E_\nabla^\sharp$ is skew-symmetric, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $h \in \mathrm{O}(T_xX) \cong \mathrm{O}(6)$ such that

$$h^{-1}E_\nabla^\sharp h = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\lambda_3 \\ \lambda_3 & 0 \end{pmatrix}.$$

In other words, we have

$$h^*E_\nabla = \lambda_1 e^1 \wedge e^2 + \lambda_2 e^3 \wedge e^4 + \lambda_3 e^5 \wedge e^6$$

for an orthonormal basis $\{e^i\}_{i=1}^6$ of T_x^*X. Then, we obtain

$$\begin{aligned} \det\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right) =& (1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda_3^2) \\ =& 1 + \sum_{i=1}^3 \lambda_i^2 + (\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) + \lambda_1^2\lambda_2^2\lambda_3^2 \\ =& 1 + |E_\nabla|^2 + \left|\frac{E_\nabla^2}{2!}\right|^2 + \left|\frac{E_\nabla^3}{3!}\right|^2. \end{aligned}$$

(2) is immediate from (1). We can also check (3), (4) easily. $\square$

We might want to consider the volume functional on a noncompact manifold. But since $v(\nabla) \geq 1$, when X is noncompact and $\mathrm{Vol}(X) = \infty$, we always have $V(\nabla) = \infty$, which is not so good.

Then define the **normalized volume functional** $V^0 : \mathcal{A}_0 \longrightarrow [0, \infty]$ by

$$V^0(\nabla) = \int_X (v(\nabla) - 1)\mathrm{vol}_g.$$

We see that

$$V^0(\nabla) = 0 \qquad \Longleftrightarrow \qquad E_\nabla = 0.$$

Then we can consider the first variation of V^0 (or V) and obtain the following.

Proposition 4.3 (The first variation). *Let $\{\nabla_t\}_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{A}_0$ be a compactly supported variation of $\nabla = \nabla_0 \in \mathcal{A}_0$ with $V^0(\nabla) < \infty$. Set*

$$a = \frac{1}{\sqrt{-1}} \frac{d}{dt}\nabla_t\bigg|_{t=0} \in \Omega^1_c = \{\textit{compactly supported 1-forms }\}.$$

Then

$$\frac{d}{dt}V^0(\nabla_t)\bigg|_{t=0} = -\langle a, H(\nabla)\rangle_{L^2}.$$

Here,

$$H(\nabla) = v(\nabla)\cdot (G_\nabla^{-1})^*\left(\sum_{j=1}^n i(G_\nabla^{-1}(e_j))D_{e_j}E_\nabla\right) \in \Omega^1(X),$$

where D is the Levi-Civita connection of g and $\{e_j\}$ is a local orthonormal frame.

Definition 4.4. We call $H(\nabla)$ the **mean curvature** of $\nabla \in \mathcal{A}_0$. $\nabla \in \mathcal{A}_0$ is said to be **minimal** if $H(\nabla) = 0$.

This proof requires a large amount of calculation. I omit the proof. For the proof, see the proof of Proposition 5.1 of

- K. Kawai, A monotonicity formula for minimal connections, arXiv:2309.11796.

Remark 4.5. We can consider the “mirror” mean curvature flow. That is, a smooth family $\{\,\nabla_t\,\}_{t\in[0,T)} \subset \mathcal{A}_0$, where $T \in (0,\infty]$, satisfies the “mirror” mean curvature flow if

$$\frac{\partial}{\partial t}\left(\frac{\nabla_t}{\sqrt{-1}}\right) = H(\nabla_t).$$

The study of this flow would be interesting. The short-time existence and uniqueness is proved in Theorem 3.7 of

- K. Kawai and H. Yamamoto, Mirror of volume functionals on manifolds with special holonomy. Adv. Math. 405 (2022), Paper No. 108515, 69 pp.

We can understand the minimality condition as follows.

For $\nabla \in \mathcal{A}_0$, define $\delta_\nabla : \Omega^k \to \Omega^{k-1}$ and $\Delta_\nabla : \Omega^k \to \Omega^k$ by

$$\delta_\nabla \alpha := -\sum_{j=1}^n i(G_\nabla^{-1}(e_j))D_{e_j}\alpha, \qquad \Delta_\nabla := d\delta_\nabla + \delta_\nabla d.$$

We can check that

- Δ_∇ is an elliptic operator.
- $E_\nabla = 0 \Longrightarrow \delta_\nabla = d^*$.

Corollary 4.6. *$\nabla \in \mathcal{A}_0$ is minimal $\Longleftrightarrow \delta_\nabla E_\nabla = 0$.*

Remark 4.7. Recall from Proposition 3.4 that the graph f is minimal if and only if $\delta_\nabla E_\nabla = 0$. Thus minimal (graphical) submanifolds correspond to minimal connections via the real Fourier–Mukai transform. In this sense, minimal connections are considered as "mirrors" of minimal submanifolds.

We see that this is a similar characterization to Yang–Mills connections: $d^* E_\nabla = 0$. Since $dE_\nabla = 0$ by the Bianchi identity (Proposition 2.7), a minimal connection ∇ satisfies $\Delta_\nabla E_\nabla = 0$.

We can make this more precise as follows.

Remark 4.8. We can show that the formal "large radius limit" of the defining equation of minimal connections ($\delta_\nabla E_\nabla = 0$) is that of Yang–Mills connections ($d^* E_\nabla = 0$). Consider the family of metrics

$$\{g_r := r^2 g\}_{r>0}.$$

Denote by $\sharp_r$ the $\sharp$ operator for g_r as before. That is,

$$E_\nabla = g_r\Big(E_\nabla^{\sharp_r}(\cdot), \cdot\Big) = r^2 g\Big(E_\nabla^{\sharp_r}(\cdot), \cdot\Big) \qquad \Longleftrightarrow \qquad E_\nabla^{\sharp_r} = \frac{1}{r^2} E_\nabla^\sharp.$$

Set

$$G_\nabla^r = \mathrm{id}_{TX} - E_\nabla^{\sharp_r} \circ E_\nabla^{\sharp_r} = \mathrm{id}_{TX} - \frac{1}{r^4} E_\nabla^\sharp \circ E_\nabla^\sharp.$$

Note that the Levi-Civita connection of g_r agrees with that of g because the Christoffel symbols are given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^n g^{k\ell} \left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right).$$

Also, note that if $\{e_j\}$ is a local orthonormal frame for g, $\{e_j/r\}$ is a local orthonormal frame for g_r. So the defining equation of minimal connections with respect to g_r is given by

$$\delta_\nabla^r E_\nabla := -\frac{1}{r^2} \sum_{j=1}^n i\big((G_\nabla^r)^{-1}(e_j)\big) D_{e_j} E_\nabla = -\frac{1}{r^2} \sum_{j=1}^n i\left(\left(\mathrm{id}_{TX} - \frac{1}{r^4} E_\nabla^\sharp \circ E_\nabla^\sharp \right)^{-1} (e_j) \right) D_{e_j} E_\nabla = 0.$$

Thus, formally taking the "large radius limit", which means the leading behaviour of $\mathcal{F}_r(\nabla)$ as $r \to \infty$, we obtain

$$d^* E_\nabla = 0.$$

This is exactly the defining equation of Yang–Mills connections. Thus it is natural to expect that minimal connections for a sufficiently large metric will behave like Yang–Mills connections.

Using this, we can show the following existence theorem.

Theorem 4.9. *Suppose further that (X, g) is compact. Then there exists a minimal connection with respect to g_r for sufficiently large $r > 0$.*

We use the fact that there is a Yang-Mills connection for a line bundle (by the Hodge theory), and the implicit function theorem.

Outline of the proof. Define a map $\mathcal{F} : [0,1] \times \mathcal{A}_0 \to d^*\Omega^2$ by

$$\mathcal{F}(s, \nabla) = -\left(\det \widetilde{G}^s_\nabla\right)^{1/4} \left(\left(\widetilde{G}^s_\nabla\right)^{-1}\right)^* i\left(\left(\widetilde{G}^s_\nabla\right)^{-1}(e_i)\right) D_{e_i} E_\nabla,$$

where

$$\widetilde{G}^s_\nabla := \mathrm{id}_{TX} - s^4 E^\sharp_\nabla \circ E^\sharp_\nabla.$$

Then

$$\mathcal{F}(s, \nabla) = \begin{cases} d^* E_\nabla & s = 0, \\ -\frac{1}{s^2} H^{1/s}(\nabla) & s \neq 0, \end{cases}$$

where $H^{1/s}(\nabla)$ is the mean curvature for $g_{1/s}$ as defined in Proposition 4.3. Then $\mathcal{F}(0, \cdot)^{-1}(0)$ is the set of Yang–Mills connections with respect to g and $\mathcal{F}(s, \cdot)^{-1}(0)$ for $s \neq 0$ is the set of minimal connections with respect to $g_{1/s}$. (We omit the explanation that the image of $\mathcal{F}$ is contained in $d^*\Omega^2$.)

By the Hodge decomposition, we see that there is a Yang–Mills connection ∇_0, i.e., an element $\nabla_0 \in \mathcal{F}(0, \cdot)^{-1}(0)$. Then we use the "implicit function theorem" to show the existence of a minimal connection for a small s.

More precisely, we first consider the derivative (linearization) $(d\mathcal{F})_{(0,\nabla_0)} : \mathbb{R} \oplus \sqrt{-1}\Omega^1 \to d^*\Omega^2$ of $\mathcal{F}$ at $(0, \nabla_0)$. We have

$$(d\mathcal{F})_{(0,\nabla_0)}(0, \sqrt{-1}b) = \frac{d}{dt}\mathcal{F}(0, \nabla_0 + t\sqrt{-1}b \cdot \mathrm{id}_L)\bigg|_{t=0} = -d^* db.$$

By the Hodge decomposition, we see that this map is surjective. Then by the "implicit function theorem", there is $\nabla \in \mathcal{A}_0$ s.t. $\mathcal{F}(s, \nabla) = 0$ for small s, which is equivalent to saying that ∇ is a minimal connection with respect to $g_{1/s}$. $\square$

We can also show the following monotonicity formula for minimal connections.

Theorem 4.10 (Monotonicity formula).
- *(X^n, g): an oriented Riemannian manifold, with $\underline{\dim X = n = 2m+1}$ and $\underline{\mathrm{Ric}(g) \geq 0}$. Fix $p \in X$.*
- *$(L, h) \to X$: a smooth complex Hermitian line bundle.*

Then there exist $a = a(n,p,g) \geq 0, 0 < r'_p < \mathrm{inj}_g(p)$, and a function $\Theta : [0,\infty) \to \mathbb{R}$ s.t. for any minimal connection ∇,

$$(0, r'_p] \to \mathbb{R}, \qquad \rho \mapsto \frac{e^{a\rho^2}}{\rho} \int_{B_\rho(p)} (v(\nabla) - 1)\mathrm{vol}_g + 2a\Theta(\rho)$$

is non-decreasing, where $B_\rho(p)$ is the geodesic ball of radius ρ centered at p.

Remark 4.11. Roughly, Theorem 4.10 says that

$$\frac{e^{a\rho^2}}{\rho^\kappa} \int_{B_\rho(p)} (v(\nabla) - 1)\mathrm{vol}_g$$

is non-decreasing for $\kappa = 1$. It is known that the value of κ is important.

- I am not sure $\kappa = 1$ the best for the monotonicity. That is, we might be able to prove the monotonicity for $\kappa > 1$.
- For Yang–Mills connections, there is an analogous monotonicity formula. In that case, κ is taken to be "scaling invariant" (in a certain sense). There are no such a property for our case.

In addition to this, if we can also prove the "ε-regularity theorem", we might study the "blowup set" of a sequence of minimal connections. (There is such an argument for Yang–Mills connections.)

Outline of the proof. • We first show the "integration by parts formula" for a minimal connection ∇:

$$\int_X (\Delta_\nabla f_1) \cdot f_2 \cdot v(\nabla)\mathrm{vol}_g = \int_X f_1 \cdot (\Delta_\nabla f_2) \cdot v(\nabla)\mathrm{vol}_g,$$

where $f_1, f_2 \in \Omega^0$, one of which is compactly supported.

• Set $f_1 = 1$, $f_2 =$ "cut off function" and compute $\Delta_\nabla f_2$.

• After some calculations, we see that the monotonically is obtained if the following is satisfied:

(1) $0 < \exists r'_p < \mathrm{inj}_g(p)$, $\forall \tau \in [0, r'_p]$,

$$n \int_{B_\tau(p)} \mathrm{vol}_g \geq \tau \frac{\partial}{\partial \tau} \int_{B_\tau(p)} \mathrm{vol}_g, \qquad \omega_n \tau^n \geq \int_{B_\tau(p)} \mathrm{vol}_g,$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$.

(2) $(\mathrm{tr} G_\nabla^{-1} - 1)v(\nabla) - n + 1 \geq 0$.

(1) is satisfied if $\mathrm{Ric}(g) \geq 0$ (relative volume comparison theorem).

(2) is an algebraic condition. It is satisfied if $\dim X = n = 2m+1$. (If $\dim X = n = 2m+1$, $E^\sharp_\nabla$ must have an eigenvalue 0. We use this.) □

For more details, see Theorem 4.15 of

- K. Kawai and H. Yamamoto, Mirror of volume functionals on manifolds with special holonomy. Adv. Math. 405 (2022), Paper No. 108515, 69 pp.

Corollary 4.12. *Let $(L, h) \longrightarrow \mathbb{R}^{2m+1}$ be a (necessarily trivial) smooth complex Hermitian line bundle over $(\mathbb{R}^{2m+1}, g_0)$, where g_0 is the standard flat metric.*

If ∇ is minimal with $V^0(\nabla) < \infty$, then ∇ is flat. (i.e. $E_\nabla = 0$.)

Proof. We use the fact that we can take $a = 0$ and $r'_p = \infty$ for $(\mathbb{R}^{2m+1}, g_0)$.

If $E_\nabla \neq 0$, $\exists p \in \mathbb{R}^{2m+1}, \exists R_0 > 0$ s.t.

$$\frac{1}{R_0}\int_{B_{R_0}(p)} (v(\nabla) - 1)\mathrm{vol}_{g_0} > 0.$$

By the monotonicity formula, for $\forall R \geq R_0$,

$$0 < \frac{1}{R_0}\int_{B_{R_0}(p)} (v(\nabla) - 1)\mathrm{vol}_g \leq \frac{1}{R}\int_{B_R(p)} (v(\nabla) - 1)\mathrm{vol}_g \longrightarrow 0 \quad (R \longrightarrow \infty),$$

which is a contradiction. □

5. Calibrated submanifolds and their mirrors

We will state a little bit about calibrated submanifolds and their mirrors.

Definition 5.1 (Harvey-Lawson, 1982)**.** Let (X^n, g) be a Riemannian manifold and $\xi \in \Omega^k(X)$ with $d\xi = 0$. ξ is called a **calibration** if for every oriented k-dimensional submanifold N

$$\xi|_N \leq \mathrm{vol}_N. \qquad \left(\Longleftrightarrow \quad \begin{array}{c} \xi(e_1, \cdots, e_k) \leq 1 \\ \text{for oriented o.n.b. } \{e_i\} \text{ of } T_xN\ (\forall x \in N). \end{array}\right)$$

N is called a **calibrated submanifold (ξ-submanifold)** if $\xi_N = \mathrm{vol}_N$.

Lemma 5.2. *Every compact calibrated submanifold N is volume-minimizing in its homology class. The volume is given topologically ($[\xi] \cdot [N]$).*

Hence calibrated submanifolds are minimal submanifolds.

Proof. Suppose that N' is any compact k-submanifold of X with $[N'] = [N] \in H_k(X, \mathbb{R})$. Then

$$\mathrm{Vol}(N) = \int_N \mathrm{vol}_N = \int_N \xi = \int_{N'} \xi \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{Vol}(N').$$

□

Example 5.3. Let (X^n, g, ω) be a Kähler manifold, where $\omega \in \Omega^2(X)$ is a Kähler form. It is known that ω and its powers (multiplied by a constant) are calibrations and calibrated submanifolds are complex submanifolds.

In other words, calibrations and calibrated submanifolds are a generalization of these.

Recall the situation of Section 3, i.e., suppose that

$$X = B^k \times T^n, \qquad X^* = B^k \times (T^n)^*.$$

If there is a calibration on X, we can impose a condition that a graph(f) of $f : B^k \longrightarrow T^n$ is a calibrated submanifold. This condition can be described in terms of the real Fourier–Mukai transform ∇, and sometimes this condition is described without torus bundle structures.

A G_2**-manifold** is defined as a 7-dimensional Riemannian manifold (X^7, g) with holonomy group $\mathrm{Hol}(g)$ contained in G_2. It is known that the metric g is Ricci-flat and there is a parallel 3-form $\varphi \in \Omega^3(X^7)$, which characterize the geometry. It is known that this 3-form φ is a calibration, and the corresponding calibrated submanifolds are called **associative submanifolds**.

We can equip $B^3 \times T^4$ with the flat G_2-structure. Then as above, we can describe the associative condition on graph(f) in terms of the real Fourier–Mukai transform ∇, and this condition is described without torus bundle structures. Then we obtain the following notion.

Definition 5.4.
- (X^7, φ, g): a G_2-manifold,
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

A Hermitian connection ∇ of (L, h) is called a **deformed Donaldson–Thomas (dDT) connection (deformed G_2-instanton)** if

$$\frac{1}{6} F_\nabla^3 + F_\nabla \wedge *\varphi = 0.$$

When $X = B^3 \times T^4$ with the flat G_2-structure, graph(f) is associative if and only if the real Fourier–Mukai transform ∇ is a dDT connection.

We can show the following.

Theorem 5.5 ("Mirror" of associator equality)**.** *Let (X^7, φ, g) be a G_2-manifold. For any $\nabla \in \mathcal{A}_0$, we have*

$$\left(1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right)^2 + \left|*\varphi \wedge F_\nabla + \frac{1}{6} F_\nabla^3\right|^2 + \frac{1}{4}|\varphi \wedge *(F_\nabla)^2|^2 = v(\nabla)^2,$$

where $v(\nabla) = \sqrt{\det\left(\mathrm{id}_{TX} + E_\nabla^\sharp\right)}$ as defined in Definition 4.1. In particular,

$$\left|1 + \frac{1}{2}\langle F_\nabla^2, *\varphi\rangle\right| \leq v(\nabla)$$

for any $\nabla \in \mathcal{A}_0$. The equality holds if and only if ∇ is dDT.

By an algebraic computation, we see that

$$*\varphi \wedge F_\nabla + \frac{1}{6} F_\nabla^3 = 0 \qquad \Longrightarrow \qquad \varphi \wedge *(F_\nabla)^2 = 0.$$

Using this, we obtain the last characterization.

By Theorem 5.5, we see the following.

Corollary 5.6. *For any dDT connection ∇, ∇ is a global minimizer of V and $V(\nabla)$ is given topologically, i.e.,*

$$V(\nabla) = \left| \int_X \left(1 + \frac{1}{2} \langle F_\nabla^2, *\varphi \rangle \right) \mathrm{vol}_g \right| = \left| \mathrm{Vol}(X) + \left(-2\pi^2 c_1(L)^2 \cup [\varphi] \right) \cdot [X] \right|$$

for any dDT connection ∇.

This is the "mirror" of the fact that every compact associative (calibrated) submanifold is homologically volume minimizing, and the volume is given topologically.

Corollary 5.7. *Suppose that L is a flat line bundle. Then any dDT connection is a flat connection. In particular, the moduli space of dDT connections is $H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z})$.*

Proof. Let ∇_0 be a flat connection (and hence ∇_0 is dDT) and ∇ be any dDT connection. Then

$$\int_X \sqrt{1 + |F_\nabla|^2 + \left| \frac{F_\nabla^2}{2!} \right|^2 + \left| \frac{F_\nabla^3}{3!} \right|^2} \mathrm{vol}_g = V(\nabla) = V(\nabla_0) = \int_X \mathrm{vol}_g,$$

which implies that $F_\nabla = 0$. □