Complex representations of finite groups
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These are lecture notes with exercises for a two week course of $12 \times 2 = 24$ hours given in 2011, 2012 and 2023 at the Nesin Matematik Köyü in Şirince, Turkey. They should be accessible to a 4th year student in mathematics; syllabus and prerequisites are described below. I wish to thank Yiğithan Tamer for his many questions and suggestions.

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Each section corresponds to a lecture of 2 hours, with the introduction fitting into § 1. The material is very standard. §§ 1–5 is one block; § 2 and § 3 may be exchanged. The exercise session § 6 can be skipped, but at the risk of losing practical understanding of phenomena. The three blocks §§ 7–8, §§ 9–10, and §§ 11–12 may be swapped freely; however § 10 uses some material from § 7. • The optional (and untaught) § 13 is a module-theoretic rephrasing of §§ 1–5. Turkish students tend to be familiar with module terminology early in their curriculum. If I were to teach the course again, I might opt for module language and introduce the group algebra as soon as § 1.

Introduction

Group theory is the natural language for symmetries and was promoted as such. As a matter of fact and contrary to modern-style expositions, groups first emerged as groups of symmetries. Some symmetries (not to be mistaken for reflections) have undisputed
geometric origin, as in Klein’s *Erlanger Programm*. Some are more combinatorial or algebraic in nature, as in Galois’ theory of polynomial equations. In either case, mathematical practice is full of groups of transformations, viz. of group actions. In the xix\textsuperscript{th} century, groups were sets of bijections preserving some structure. ‘Abstract’ groups were introduced later, around Burnside’s time, in order to achieve some unification and for intrinsic interest.

The purpose of representation theory is to return to ‘concrete’ groups, viz. groups acting somewhere. Permutation group theory embeds group theory into combinatorics, thus yielding counting arguments. But (linear) representation theory even embeds group theory into linear algebra. There one conveniently relies on geometric intuition.

Just like a group action of $G$ is simply a morphism $G \to \text{Sym}(X)$ for some set $X$, a linear representation is simply a morphism from $G$ to $\text{GL}(V)$ for some vector space. What is remarkable is that to some extent, $G$ ‘is’ the class of its representations. (There is almost some phenomenological lesson here: an abstract object is entirely determined by its concrete manifestations.) Equally importantly, complex representations of finite groups are themselves determined by some number-theoretic functions called characters. As a result, the amount of group-theoretic information encoded in characters is beyond first expectations; the strength of Frobenius’ character theory is a miracle.

The course will describe this theory, with some of its most classical and celebrated applications to ‘pure’ finite group theory. All the material here is extremely standard; neither the exposition nor the choice of contents has any claim to originality. (See the Further reading section for deeper sources.) Before you start with the notes, let me recommend a lovely survey by a major contributor to the topic.\footnote{C. Curtis, *Representation theory of finite groups: From Frobenius to Brauer*. Mathematical Intelligencer 14, No. 4, pp. 48–57, 1992.}

**Prerequisites**

The class is for advanced undergraduate or graduate students.

**Algebraic number theory:** Almost none. The characteristic of a field; algebraically closed fields. It is safe to assume $\mathbb{K} = \mathbb{C}$ everywhere. (See ‘Note on fields’ below.)

**Group theory:** Prerequisites corresponding to a full first course in pure group theory. Groups, subgroups, normal subgroups and quotient groups; morphisms and factorisation; cosets and Lagrange’s theorem on orders; conjugation and centralisers; group actions, stabilisers, orbits. Advanced topics will occasionally involve semidirect products.

I shall use $x' = y^{-1}xy$ for conjugacy; a typical conjugacy class will be denoted by $y$. I reserve $c$ for left cosets, viz. subsets of the form $c = aH = \{ah : h \in H\}$ whenever $H \leq G$ is a subgroup.

**Linear algebra:** Finite-dimensional vector spaces; linear maps, eigenvalues and trace of an endomorphism; projectors; linear forms, dual space; bilinear forms, non-degenerate bilinear forms, symmetric forms, Hermite-symmetric forms and complex scalar products. The course tends to avoid bases and matrices but prefers ‘intrinsic’ arguments, so an abstract course in linear algebra is a prerequisite.

**Module theory:** Terminology helps, but no knowledge of module theory is required. Familiarity with the tensor product is not required, as it will be briefly discussed.
Note on fields. Most arguments need assumptions both on the characteristic of $\mathbb{K}$ and its algebraic closedness.

Technically the conjunction (characteristic o and algebraically closed) defines the class $\text{ACF}_{\text{o}}$ of algebraically closed fields of characteristic o. Two examples are the field of complex numbers $\mathbb{C}$ and the field of algebraic numbers, viz. the algebraic closure $\overline{\mathbb{Q}}$ of the rationals inside the complex field. Characteristic o theory naturally takes place over number fields, viz. over finite extensions of $\mathbb{Q}$. Now every number field embeds into $\overline{\mathbb{Q}}$, so into $\mathbb{C}$. (Moreover, $\mathbb{C}$-vector spaces bear additional 'Hermite structure', also known as complex scalar products. This is the reason why working over $\mathbb{C}$ is so efficient.)

The beginner can safely assume $\mathbb{K} = \mathbb{C}$ throughout. It is however a good exercise to understand what assumptions are needed in each theorem, so we discuss characteristics a little.

Most of the theory works similarly provided char $\mathbb{K}$ is not a prime factor of $|G|$; we call this $\mathbb{K}$ has coprime characteristic with respect to $G$. One could introduce for each finite $G$ the class $\text{ACF}_{[G]}$ of algebraically closed fields of characteristic coprime to $|G|$, viz. of good fields. For brevity, a good field will be an algebraically closed field of coprime characteristic. Hence 'algebraically closed of characteristic o' implies good.

Definition. Let $G$ be a finite group and $\mathbb{K}$ be a field.

- $\mathbb{K}$ has coprime characteristic with respect to $G$ if (char $\mathbb{K} = o$ or char $\mathbb{K}$ does not divide $|G|$).
- $\mathbb{K}$ is good if it is algebraically closed of coprime characteristic.

Remarks.

- Coprimality makes sense only for finite $G$; so does goodness. Almost nothing remains of the theory for infinite $G$. There is no general representation theory of abstract infinite groups.
- Because the characteristic of a field is o or a prime number, this is equivalent to: char $\mathbb{K}$ is o or does not divide the exponent exp $G$, which is the least common multiple of orders of elements of $G$.
- The fields $\overline{\mathbb{Q}}$, $\mathbb{C}$ are universally good (viz. independently of $G$).

1 Representations

Abstract. This section is mostly terminology. § 1.1 introduces the main objects: representations. These simply consist in an action on some vector space. § 1.2 discusses subrepresentations, and we stress the importance of irreducible representations. Finally § 1.3 describes morphisms of representations.

1.1 Representations

1.1.1. Definition. Let $G$ be a group and $\mathbb{K}$ be a field.

- A linear representation of $G$ in a $\mathbb{K}$-vector space $V$ is a morphism $\rho: G \rightarrow \text{GL}(V)$.
- A linear representation of $G$ over $\mathbb{K}$, or $\mathbb{K}$-linear representation, is a pair $(V, \rho)$ as above; one often omits $V$ or $\rho$ from notation.

(Omitting $V$ is less ambiguous than omitting $\rho$. Common practice can do either.)
1.1.2. Remarks.

- In the notation above, let \( g \in G \) and \( v \in V \). Then \( \rho(g) \in \text{GL}(V) \), so \( \rho(g)(v) \in V \).

As alternatives to this clumsy notation, one may write:

  - \( \rho(g) \cdot v \) (useful if there are several representations),
  - \( g \ast v \) (useful to distinguish operators from vectors),
  - \( g \cdot v \) or simply \( gv \) (useful to save time).

- By definition of a morphism, one has \( \rho(1) = \text{Id}_V \) and \( \rho(gh^{-1}) = \rho(g) \circ \rho(h)^{-1} \).

- Tradition calls degree of the representation the cardinal number \( \deg \rho = \dim V \).

- A linear representation \( \rho \) of \( G \) induces an injective linear representation of \( G/\ker \rho \).

- Tradition calls faithful an injective representation, viz. one with \( \ker \rho = \{1\} \).

1.1.3. Examples.

- The trivial representation is:

  \[
  \text{triv}: \quad G \to \text{GL}(\mathbb{K})
  \]

  \[
  g \mapsto 1,
  \]

  with dimension 1.

- Let \( V \) be the \( \mathbb{K} \)-vector space with dimension \( |G| \) and basis \( B = \{ e_g : g \in G \} \). For \( g \in G \) we define \( \text{reg}(g) \) on the basis \( B \) by letting \( \text{reg}(g)(e_h) = e_{gh} \). Then we extend linearly, meaning we let:

  \[
  \text{reg}(g)(\sum \lambda_h e_h) = \sum \lambda_h e_{gh}.
  \]

  This defines the regular representation:

  \[
  \text{reg}: \quad G \to \text{GL}(V)
  \]

  \[
  g \mapsto \text{reg}(g).
  \]

  (Check that one does have \( \text{reg}(g,g^{-1}) = \text{reg}(g) \circ (\text{reg}(g))^{-1} \) in \( \text{GL}(V) \)).

  From § 13 on, we shall drop unnecessary symbols and simply write \( g \) for \( e_g \).

- Technically, the above is the left-regular representation. (See exercise 1.4.4.)

- The regular representation is a special case of a more general construction. Let \( G \) act on some set \( X \) by \( g \ast x \). Let \( V \) be the vector space with dimension \#\(X\) and basis \( B = \{ e_x : x \in X \} \). Now define \( \text{perm}(g) \) on the basis \( B \) by:

  \[
  \text{perm}(g)(e_x) = e_{gx},
  \]

  then extend linearly. This defines the permutation representation associated to the permutation group \( (G, X) \). Its kernel is exactly the kernel of the action.

- The regular representation is thus the permutation representation associated to the left-regular, ‘Cayley’ group action of \( G \) on itself: \( g \ast x = gx \).

  It is injective. Indeed, \( \text{reg}(g) = \text{Id}_V \) implies \( \text{reg}(g)(e_i) = \text{Id}(e_i) = e_i \) while \( \text{reg}(g)(e_i) = e_{gi} \). So \( g = 1 \) and \( \ker \text{reg} = \{1\} \).

- Not all representations are permutation representations, so the topic does not reduce to group actions.

1.1.4. Remarks (even more general representations).
Onemayrepresentotheralgebraicstructures;forexample,arepresentationofan (associative) ring $R$ in avectorspace $V$ issimplyaringmorphism $R \to \text{End}(V)$. Actually, $\mathbb{K}$-linear representations of a group $G$, correspond to representations of the ‘group algebra’ $\mathbb{K}[G]$, by taking $\rho: G \to \text{GL}(V)$ to the linear extension $\hat{\rho}: \mathbb{K}[G] \to \text{End}(V)$. (This is discussed in §13.)

Onemayalsorepresentalgebraicstructuresinmoregeneralmodulesthanvectorspaces. Forinstance, arepresentationofagroup $G$ in anabeliangroup $A$ issimply anmorphism $G \to \text{Aut}(A)$. (This can be interesting, and bring new phenomena, if $A$ is for instance the torsion subgroup of the circle group $\mathbb{S}^1 = \text{SO}_2(\mathbb{R})$, which is no vector space.) Likewise, onemayrepresentassociativerings,orevenLierings, droppinglinearity.

Linear algebra gives intuition and tools, and linear representations of a given group $G$ already encode much information about it.

The purpose of this course is to give general notions on linear representations of finite groups in finite-dimensional vector spaces over good fields, such as $\mathbb{C}$. This is a well-chartered territory, but it has striking applications. The following basic lemma is used throughout.

1.1.5. Lemma. Let $G$ be a finite group and $\mathbb{K}$ be a good field. Let $\rho: G \to \text{GL}(V)$ be a $\mathbb{K}$-linearspace $W \leq V$ which is also $G$-invariant, viz. $(\forall g \in G)(\forall w \in W)(g \cdot w \in W)$.

This is the same as a $\mathbb{K}[G]$-submodule; see §13.2.)

A nonzero representation $V$ is irreducible if the only $G$-invariant subspaces are $\{0\}$ and $V$. (This is the same as simplicity as a $\mathbb{K}[G]$-module; see §13.1.)

Notice that by convention, $\{0\}$ is not irreducible (similar to ‘1 is not a prime’).

1.2 Subrepresentations and irreducibility

We discuss subobjects.

1.2.1. Definition. Let $G$ be a group and $\mathbb{K}$ be a field. Let $(V, \rho)$ be a $\mathbb{K}$-linear representation.

• A subrepresentation is a $\mathbb{K}$-linear subspace $W \leq V$ which is also $G$-invariant, viz. $(\forall g \in G)(\forall w \in W)(g \cdot w \in W)$.

This is the same as a $\mathbb{K}[G]$-submodule; see §13.2.)

• A nonzero representation $V$ is irreducible if the only $G$-invariant subspaces are $\{0\}$ and $V$.

This is the same as simplicity as a $\mathbb{K}[G]$-module; see §13.1.)

1.2.2. Examples.
1. If $A$ is an abelian group and $\mathbb{K}$ is an algebraically closed field, then every finite-dimensional, $\mathbb{K}$-linear representation of $A$ is actually 1-dimensional. This is an important fact (exercise 1.4.3).

2. This need not hold over arbitrary $\mathbb{K}$. Let $G = \text{SO}_2(\mathbb{R})$ be the real unit circle, acting by rotations on $V = \mathbb{R}^2$. This representation is irreducible because no vector line is invariant under rotations.

Irreducible representations are the building blocks, the ‘atoms’ of more elaborate representations (accordingly thought of as molecules). This analogy underlies the entire theory over good fields and is developed in §§ 3–4.

1.2.3. Lemma. Let $G$ be a finite group and $\mathbb{K}$ be a field.

(i) Every irreducible linear representation is finite-dimensional.

(ii) Every nonzero linear representation contains an irreducible representation.

Proof.

(i) Let $V$ be irreducible. Let $v \in V \setminus \{0\}$. Now the set $X = \{g \cdot v : g \in G\}$ is finite, hence spans a finite-dimensional subspace $W \leq V$. Any $g \in G$ acts on $X$, so it leaves $W$ invariant. Hence $W \leq V$ is a subrepresentation. Since $0 \neq v \in W$, irreducibility implies $V = W$, which is finite-dimensional.

(ii) Let $V \neq 0$ be arbitrary. Here again, $V$ contains a finite-dimensional subrepresentation $W$. Now a finite-dimensional subrepresentation of minimal dimension is irreducible.

1.2.4. Remarks.

- Some infinite groups do not have a non-trivial, finite-dimensional representation (even reducible). See exercise 1.4.5.
- Some groups, and even some finite groups, do not have an injective irreducible representation. See exercise 1.4.6.
- Still, every group has an injective representation—for instance the regular one.

1.2.5. Remark (quotient representations). If $V$ is a $\mathbb{K}$-linear representation and $W \leq V$ a subrepresentation, then the quotient group $V/W$ naturally bears the structure of a $\mathbb{K}$-linear representation of $G$, called quotient representation.

Due to Maschke's splitting phenomenon discussed in § 3.2, quotient representations turn out to be avoidable in a first course, very much like quotient vector spaces are usually omitted from basic treatments of linear algebra.

1.3 Morphisms and isomorphisms of representations

We discuss arrows.

1.3.1. Definition. Let $G$ be a group and $\mathbb{K}$ be a field. Let $(V_1, \rho_1)$ and $(V_2, \rho_2)$ be two $\mathbb{K}$-linear representations of $G$. 

6
• A morphism of representations, also called a $G$-covariant morphism, is a function $f: V_1 \to V_2$ which is both $K$-linear and compatible with $G$, viz. in obvious notation:

$$f(\rho_1(g) \cdot v_1) = \rho_2(g) \cdot f(v_1).$$

In lighter notation, this rewrites as $f(g \cdot v_1) = g \cdot f(v_1)$.

• This is the same as a morphism of $K[G]$-modules. (See § 13.2.)

• We let $\text{Hom}_{K[G]}(\rho_1, \rho_2)$ be the set of morphisms of representations.

• More casual notations are $\text{Hom}_{K[G]}(V_i, V_j)$, or $\text{Hom}_G(\rho_1, \rho_2)$, or $\text{Hom}_G(V_i, V_j)$.

• If $V_i = V_j = V$ (implicitly we request $\rho_1 = \rho_2 = \rho$; see remarks 1.3.3), we simply write $\text{End}_{K[G]}(\rho)$. (More casual: $\text{End}_G(\rho)$ or $\text{End}_G(V)$.)

• An alternative notation could be $C_{\text{End}_G(V)}(G)$, because it consists of those $K$-linear maps $f \in \text{End}_K(V)$ which commute with the 'Hom' action of $G$, in the sense of § 2.4.

• An isomorphism is a bijective morphism. Notice that the inverse of an isomorphism is an isomorphism.

• Tradition calls equivalence an isomorphism of representations.

1.3.2. Remark. A $K$-linear map $f: V_i \to V_j$ is $G$-covariant iff for all $g \in G$ the following diagram is commutative, viz. the two possible compositions define the same morphism:

\[
\begin{array}{ccc}
V_i & \xrightarrow{f} & V_j \\
\downarrow{\rho_i(g)} & & \downarrow{\rho_j(g)} \\
V_i' & \xrightarrow{f} & V_j'.
\end{array}
\]

1.3.3. Remarks.

• $\text{Hom}_{K[G]}(V_i, V_j)$ is a slightly ambiguous notation: what matters is not $V_i$, but truly $(V_i, \rho_i)$. Hence $\text{Hom}_{K[G]}(\rho_1, \rho_2)$ is clearer but one has to accept 'arrows between arrows'.

• In more modern notation, $(\rho_1 \to \rho_2)$ would be in order, meaning: those arrows from $\rho_1$ to $\rho_2$, being well understood that $\rho_i, \rho_2$ live in the category of $K[G]$-modules. Another option is $(\rho_1 \to \rho_2: K[G]\text{-Mod})$, for those arrows in the category $K[G]\text{-Mod}$.

1.3.4. Remark. Isomorphisms are classically denoted by $\cong$. Since there will be a constant tension between isomorphisms of underlying vector spaces, and isomorphisms of representations (which are stronger), we prefer explicit notation.

The notion of an isomorphism is relative to a category. For isomorphisms of $K$-vector spaces, we write $V_i \cong V_j$ $[K\text{-Mod}]$ ('isomorphism of $K$-modules'). For isomorphisms of $K$-linear representations of $G$, we write $V_i \cong V_j$ $[K[G]\text{-Mod}]$ ('isomorphism of $K[G]$-modules'). (See § 13.)

1.3.5. Notation. Let $G$ a group and $K$ be a field. Let $\text{Irr}_K(G)$ denote the class of the irreducible, $K$-linear representations of $G$ up to isomorphism.

Truly elements of $\text{Irr}_K(G)$ are not representations, more isomorphism classes.
1.3.6. Remarks.

- In general, $\text{Irr}_\mathbb{K}(G)$ need not be finite. But if $G$ is finite then so is $\text{Irr}_\mathbb{K}(G)$. This is proved in exercise 1.4.7.

- Actually if $G$ is finite and $\mathbb{K}$ is good then $\# \text{Irr}_\mathbb{K}(G)$ is the number of conjugacy classes in $G$, as proved in Theorem 5.1.1. In particular, $\# \text{Irr}_\mathbb{C}(G) = \# \text{Conj}(G)$.

- For algebraically closed $\mathbb{K}$ of characteristic dividing $|G|$, the number $\# \text{Irr}_\mathbb{K}(G)$ takes another value found by Brauer\(^2\). For non-algebraically closed $\mathbb{K}$, matters are more involved.

1.4 Exercises

1.4.1. Exercise. Let $G = \text{Sym}(3)$, which is generated by (12) and (123). Let $V \simeq \mathbb{C}^2$ have basis $\mathcal{B} = (e_1, e_2)$. Now let $\rho((12))$ swap $e_1$ and $e_2$, while $\rho((123))$ does the following:

$$\rho((123))(e_1) = e_2 \quad \text{and} \quad \rho((123))(e_2) = -e_1 - e_2.$$ 

1. Prove that it defines a linear representation.

2. Write the matrix $\text{Mat}_B \rho(g)$ for every $g \in G$.

1.4.2. Exercise. Let $G = \text{Sym}(n)$ be the symmetric group over $n$ symbols.

1. The sign representation of $G$ is:

$$\varepsilon: \begin{array}{c} G \\ \rightarrow \text{GL}(\mathbb{K}^n) \end{array} \quad \begin{array}{c} g \\ \rightarrow \varepsilon(g) \end{array},$$

where $\varepsilon(g)$ is the usual signature of $g$. Show that it is a representation.

2. Let $\rho: G \rightarrow \text{GL}(V)$ be a $\mathbb{K}$-linear representation. Show that $\rho'(g) = \varepsilon(g) \cdot \rho(g)$ is another $\mathbb{K}$-linear representation. (More generally see tensor representations, §2.3.)

1.4.3. Exercise.

1. Let $A$ be a finite, abelian group and $\rho \in \text{Irr}_\mathbb{C}(A)$. Show that $\dim \rho = 1$. Hint: eigenvalue. Does this generalise to nilpotent groups?

2. Let $A$ be an abelian subgroup of a finite group $G$. Let $\rho \in \text{Irr}_\mathbb{C}(G)$. Show that $\dim \rho \leq |G: A|$.

3. Characterise which finite groups have a non-trivial, complex, 1-dimensional representation $\rho \neq \text{triv}$. Did you use properties of the field?

1.4.4. Exercise. The right-regular representation is defined as follows. Let $V$ have basis $\{e_g : g \in G\}$. Now let $\text{reg}^{op}(g)(e_h) = e_{h \cdot g^{-1}}$, and extend linearly. Prove that $\text{reg}$ and $\text{reg}^{op}$ are isomorphic.

1.4.5. Exercise. Let $G$ be a group.

1. Prove that the following are equivalent:

(i) if \( X \subseteq G \) is any subset, there are finitely many \( x_1, \ldots, x_n \in X \) such that \( C_G(X) = C_G(x_1, \ldots, x_n) \);
(ii) every ascending chain of centralisers \( C_G(X_i) \leq C_G(X_j) \leq \ldots \) is stationary;
(iii) every descending chain of centralisers \( C_G(X_i) \geq C_G(X_j) \geq \ldots \) is stationary.

Hint: \( CCC = C \).

2. A linear group is a subgroup of some \( \text{GL}_n(\mathbb{K}) \) for some integer \( n \) and some field \( \mathbb{K} \). Prove that every linear group satisfies the above condition. Hint: use \( M_n(\mathbb{K}) \).

(*) 3. Let \( \text{Alt}(\mathbb{N}) \) be the set of permutations of \( \mathbb{N} \) with finite support and signature \( 1 \). Deduce that every finite-dimensional representation of \( \text{Alt}(\mathbb{N}) \) is trivial.

1.4.6. Exercise. Let \( N \cong C_4 \) be generated by \( e_1, e_2, e_3, e_4 \). Now let \( \{ \sigma \} = C_3 \) act on \( N \) as follows:
\[
e_1^\sigma = e_2, \quad e_2^\sigma = e_1 e_2, \quad e_3^\sigma = e_4, \quad e_4^\sigma = e_3 e_4.
\]

Let \( G = N \rtimes \{ \sigma \} \). Show that \( G \) has no injective irreducible representation over \( \mathbb{C} \). Hint: diagonalise \( N \) simultaneously.

Note. Characterisation of finite groups admitting a injective irreducible representation is a classical topic.\(^3\)

1.4.7. Exercise. Let \( G \) be a finite group, and let \( \mathbb{K}[G] \) denote the left-regular representation.

1. Prove that every irreducible representation \( V \) is isomorphic to some \( V' \leq \mathbb{K}[G] \). Hint: fix a non-zero linear form \( \varphi \in V^* \) and consider \( f(v) = \sum_{g \in G} \varphi(g^{-1} v)e_g \).

2. Suppose that \( V_1, \ldots, V_r \leq \mathbb{K}[G] \) are pairwise non-isomorphic irreducible representations. Prove that \( \sum V_i = \bigoplus V_i \) is a direct sum. Hint: if \( V_0 \leq \bigoplus V_i \), consider the projectors \( \pi_i \); \( V_0 \overset{\pi_i}{\rightarrow} V_i \) onto the \( i \)th coordinate.

3. Deduce that \( G \) has finitely many irreducible representations up to isomorphism.

Note. With more tools (and provided \( \text{char} \; \mathbb{K} \nmid |G| \)), there are other arguments relying on the Artin-Wedderburn theorem applied to the algebra \( \mathbb{K}[G] \).

2 Algebraic constructions

Abstract. We build new representations from existing ones. Direct sum representations (§ 2.1) are easily understood. Dual representations (§ 2.2) offer the opportunity to return to algebraic duality and dual bases in finite-dimensional spaces. Tensor representations (§ 2.3) endow the vector tensor product of two spaces with an action of \( G \). Last, Hom-representations (§ 2.4) connect to dual and tensor constructions. In general, irreducibility is lost.

This section contains almost no representation theory except a couple of definitions (dual representation, tensor representation, Hom representation). There are no assumptions on \( \mathbb{K} \). Throughout we pay attention to traces as they will underlie character theory.

2.1 Direct sum representation

Direct sum space. The notion of a direct sum space is expected to be familiar.

2.1.1. Definition. Let \( V_1 \) and \( V_2 \) be two vector spaces over the same field. Their (external) direct sum is the vector space \( V_1 \oplus V_2 \) of pairs \( (v_1, v_2) \), equipped with componentwise linear structure.

2.1.2. Remarks.

- \( V_1 \) and \( V_2 \) naturally embed into \( V_1 \oplus V_2 \). There, their internal direct sum equals \( V_1 \oplus V_2 \).
- In particular if \( V_1 \) has basis \( B_1 \) and \( V_2 \) has basis \( B_2 \), then \( V_1 \oplus V_2 \) has basis \( B_1 \cup B_2 \).
- Not to be mistaken with the direct product space—though isomorphic as long as only finitely many vector spaces are involved.

Direct sum representation. The simplest possible construction is as follows.

2.1.3. Definition. Let \( (V_1, \rho_1) \) and \( (V_2, \rho_2) \) be two representations of the same group over the same field. The direct sum representation is \( (V_1 \oplus V_2, \rho_1 \oplus \rho_2) \) given by:

\[
(\rho_1 \oplus \rho_2)(g)|_{V_1} = \rho_1(g) \quad \text{and} \quad (\rho_1 \oplus \rho_2)(g)|_{V_2} = \rho_2(g).
\]

This means that \( g \) acts on \( V_1 \oplus V_2 \) componentwise; in finite dimension, one may use ‘block matrices’. The following is therefore obvious.

2.1.4. Remark. Let \( V_1 \) and \( V_2 \) be finite-dimensional representations of \( G \). Then for \( g \in G \) one has \( \text{tr}(\rho_1 \oplus \rho_2)(g) = \text{tr} \rho_1(g) + \text{tr} \rho_2(g) \).

2.2 Dual representation

Dual space. We first review some properties of the dual space of a vector space, with no reference to representation theory.

2.2.1. Definition. Let \( V \) be a \( \mathbb{K} \)-vector space.

- A linear form on \( V \) is a linear map \( \varphi : V \to \mathbb{K} \).
- The dual space of \( V \) is the space \( V^* \) of all linear forms on \( V \), equipped with the following linear structure in obvious notation:

\[
- (\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v),
- (\lambda \varphi)(v) = \lambda \varphi(v).
\]

2.2.2. Remark (duality pairing). There is a ‘pairing’ \( V^* \times V \to \mathbb{K} \), given by \( \langle \varphi | v \rangle = \varphi(v) \).

Now if \( f : V \to V \) is a linear endomorphism, it induces \( f^* : V^* \to V^* \) defined by:

\[
f^*(\varphi) = \varphi \circ f.
\]

Notice that \( \langle f^*(\varphi) | v \rangle = \varphi \circ f(v) = \langle \varphi | f(v) \rangle \).

2.2.3. Lemma (and definition: dual basis). Let \( V \) be a finite-dimensional vector space.

(i) \( \dim V^* = \dim V \).
(ii) If \(B = \{e_1, \ldots, e_n\}\) is a basis of \(V\), let \(e_i^* \in V^*\) be the linear form such that \(e_i^*(e_j) = \delta_{ij}\). Then \(B^* = \{e_1^*, \ldots, e_n^*\}\) is a basis of \(V^*\), called the dual basis.

(iii) Let \(f \in \text{End}(V)\). In the notation above, \(\text{Mat}_B f^* = (\text{Mat}_B f)^t\).

(iv) In the notation above, \(\text{tr} f = \text{tr} f^*\).

Proof.

(i) is a consequence of (ii).

(ii) We prove both linear independence and generation. Suppose \(\sum_{i=1}^n \lambda_i e_i^* = 0\), in obvious notation. Then evaluating at each \(e_j\) gives \(\lambda_j = 0\): whence linear independence.

Now let \(\varphi \in V^*\). For \(i \in \{1, \ldots, n\}\), let \(\lambda_i = \varphi(e_i)\). Then notice that \(\varphi - \sum \lambda_i e_i^*\) vanishes on \(B\), so on \(V\). Thus \(\varphi = \sum \lambda_i e_i^*\), which proves generation. Hence \(B^*\) is a basis of \(V^*\).

(iii) Say \(\text{Mat}_B f = M = (m_{ij})\), so that \(\text{Col}_j \text{Mat}_B f = \text{Coord}_B f(e_j)\), viz. \(f(e_j) = \sum_i m_{ij} e_i\). Let \(v = \sum_i \lambda_i e_i\). Compute as follows:

\[
f^*(e_j^*)(v) = e_j^*(f(v)) = (e_j^* \circ f)(\sum_i \lambda_i e_i) = e_j^*(\sum_i \lambda_i f(e_i)) = e_j^*(\sum_i \lambda_i \sum_k m_{kj} e_k) = \sum_i \lambda_i \sum_k m_{kj} e_j^*(e_k) = \sum_i \lambda_i m_{ji} = \sum_{i,j} (M')_{ij} e_i^*(v),
\]

whence \(f^*(e_j^*) = \sum_i (M')_{ij} e_i^*\). This is our claim.

(iv) Obvious from (iii).

\[
2.2.4. \text{Remark.} \text{ This no longer holds if dim } V \text{ is infinite. One can still define linear forms } e_i^*, \text{ which remain linearly independent, but they no longer generate } V^*.
\]

**Dual representation.** We now put a \(G\)-structure on the dual vector space of a representation of \(G\), in the most natural way.

**2.2.5. Definition.** Let \((V, \rho)\) be a \(k\)-linear representation of \(G\). Its dual representation is \((V^*, \rho^*)\), where:

\[
\rho^*(g)(\varphi) = \varphi \circ \rho(g)^{-1}.
\]

In alternate notation, this rewrites \((g \cdot \varphi)(v) = \varphi(g^{-1}v)\).

**2.2.6. Remarks.**
In this course we only tensor over \( \mathbb{K} \), never over \( \mathbb{K}[G] \).

**Tensor product of vector spaces.** Informally speaking, the tensor product converts bilinear maps to linear maps.

**Proposition.** Let \( V_1, V_2 \) be \( \mathbb{K} \)-vector spaces over the same field \( \mathbb{K} \). Then there is a (unique) initial pair \( (W, \phi) \) where \( W \) is a \( \mathbb{K} \)-vector space and \( \phi : V_1 \times V_2 \to W \) is bilinear.

This means that there is a unique pair \( (W, \phi) \) as above such that for any other pair \( (W', \phi') \), one may uniquely factorise \( \phi' = h \circ \phi \).

\[
\begin{array}{c}
V_1 \times V_2 \\
\downarrow \phi' \\
W' \\
\downarrow \phi \\
W \\
\end{array}
\]

**Proof.** Say \( V_1 \) has basis \( B_1 \). Let \( C = B_1 \times B_2 \); let \( W \) have basis indexed by \( C \). Map \( (b_1, b_2) \in C \) to the corresponding vector in \( W \) and extend bilinearly. This defines \( \phi \).

We have to check that \( (W, \phi) \) meets the requirements. So let \( (W', \phi') \) be another pair. On \( \beta(b_1, b_2) \) we let \( h(\beta(b_1, b_2)) = \beta'(b_1, b_2) \). This is well-defined. Then we extend \( h \) linearly. So \( \beta' \) factor through \( \beta \). Moreover we had no other choice.

We now check that \( (W, \phi) \) is unique up to isomorphism. Indeed, if \( (W', \phi') \) is another initial pair, then there are unique \( h, h' \) such that \( \beta = h' \circ h \circ \beta \). But \( \beta = \text{Id}_W \circ \beta \) would have worked as well, so \( h' \circ h = \text{Id}_W \). Likewise, \( h \circ h' = \text{Id}_{W'} \). So \( h \) and \( h' \) are linear isomorphism and we are done.

**Definition.** In the notation above, one writes \( W = V_1 \otimes \mathbb{K} V_2 \) and \( \beta(v_1, v_2) = v_1 \otimes v_2 \). This is called the tensor product of \( V_1 \) and \( V_2 \) over \( \mathbb{K} \).

Of course the tensor product is more an isomorphism type than a given realisation; in particular one may always pick bases \( B_1 \) and \( B_2 \) more adapted to specific problems.
2.3.3. Remarks.

- Write \( B_1 \otimes B_2 = B_1 \times B_2 \). This is a basis of \( V_1 \otimes V_2 \). In particular, dim\((V_1 \otimes V_2) = \operatorname{dim} V_1 \cdot \operatorname{dim} V_2 \).

- Every element of \( V_1 \otimes V_2 \) is therefore a unique linear combination of basic elements \( b_1 \otimes b_2 \). However, it is useful to forget about bases. Thus every element is a linear combination of ‘elementary tensors’ \( v_1 \otimes v_2 \), but not every element of \( V_1 \otimes V_2 \) is an elementary tensor. (Physicists call this phenomenon ‘intrication.’)

- A remark on the construction. The general notion of a tensor product over a ring \( R \) is more involved and requires some factorisation. Since vector spaces are free modules, our construction is (fortunately) much simpler than the general module-theoretic one.

Tensor representation. We now equip the vector tensor product of representations with the structure of a representation (viz. a \( G \)-action).

2.3.4. Definition. Let \( G \) be a group and \( \mathbb{K} \) be a field. Let \(( V_1, \rho_1), (V_2, \rho_2) \) be two \( \mathbb{K} \)-linear representations. The tensor representation is \(( V_1 \otimes \mathbb{K} \otimes V_2, \rho_1 \otimes \rho_2) \) where:

\[
(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = (gv_1 \otimes gv_2).
\]

2.3.5. Lemma. \( \operatorname{tr}(\rho_1 \otimes \rho_2)(g) = (\operatorname{tr} \rho_1(g)) \cdot (\operatorname{tr} \rho_2(g)) \).

Proof. Let \( B_1 = \{ e_1, \ldots, e_p \} \) be a basis of \( V_1 \) and \( B_2 = \{ f_1, \ldots, f_q \} \) be one of \( V_2 \); then \( \{ e_i \otimes f_k \} \) is one of \( V_1 \otimes V_2 \). By definition there are matrices \( M = (m_{i,j}) \) and \( N = (n_{k,\ell}) \) such that:

\[
\rho_1(g) \cdot e_j = \sum_i m_{i,j} e_i \quad \text{and} \quad \rho_2(g) \cdot f_\ell = \sum_k n_{k,\ell} f_k.
\]

Thus:

\[
(\rho_1 \otimes \rho_2)(g) \cdot (e_j \otimes f_\ell) = \sum_{i,k} m_{i,j} n_{k,\ell} (e_i \otimes f_k).
\]

Summing diagonal terms, we find:

\[
\operatorname{tr}(\rho_1 \otimes \rho_2)(g) = \sum_{j,\ell} m_{j,j} n_{\ell,\ell} = (\sum_j m_{j,j})(\sum_\ell n_{\ell,\ell}) = (\operatorname{tr} \rho_1(g))(\operatorname{tr} \rho_2(g)). \quad \square
\]

2.4 Hom-representation

The definition.

2.4.1. Definition. Let \( V_1, V_2 \) be \( \mathbb{K} \)-vector spaces. Let \( \operatorname{Hom}_{\mathbb{K}}(V_1, V_2) \) be the space of linear maps \( V_1 \to V_2 \), equipped with the following linear structure in obvious notation:

- \((f + g)(v_1) = f(v_1) + g(v_1)\),
- \((\lambda f)(v_1) = \lambda f(v_1)\).

Thus \( V^* = \operatorname{Hom}_{\mathbb{K}}(V, \mathbb{K}) \).
2.4.2. Definition. Let \((V_1, \rho_1), (V_2, \rho_2)\) be two \(\mathbb{K}\)-linear representations of \(G\). The Hom-representation is \((\text{Hom}_\mathbb{K}(V_1, V_2), \rho_{\text{Hom}})\) where:

\[
\rho_{\text{Hom}}(g)(f) = \rho_1(g) \circ f \circ \rho_2(g)^{-1} = g \cdot f \cdot g^{-1}.
\]

2.4.3. Remark. A \(\mathbb{K}\)-linear homomorphism \(f : V_1 \to V_2\) is fixed under the Hom-action of \(G\) if and only if \((\forall g)(f \circ \rho_1(g) = \rho_2(g) \circ f)\) if and only if \(f\) is a \(\mathbb{K}[G]\)-morphism. In symbols,

\[
C_{\text{Hom}_\mathbb{K}(V_1, V_2)}(G) = \text{Hom}_{\mathbb{K}[G]}(V_1, V_2).
\]

This simple observation is crucial in character theory (§ 5.2).

Hom and tensors. Let \(f : V_1 \to V_2\) be linear between finite-dimensional spaces. Suppose \(V_1\) has basis \(B = \{e_1, \ldots, e_n\}\). Then for \(x \in V_1\), one may write \(x = \sum \lambda_j e_j\). Now \(e_j^*(x) = \lambda_j\), so \(x = \sum_j e_j^*(x)e_j\). Notice that this amounts to writing, as functions:

\[
\text{Id} = \sum_j e_j^* e_j.
\]

Furthermore \(f(x) = f\left(\sum_j e_j^*(x)e_j\right) = \sum_j e_j^*(x)f(e_j)\). Thus as functions from \(V_1\) to \(V_2\) we have:

\[
f = \sum_j e_j^* f(e_j).
\]

The proper place to deal with the right-hand is the tensor space \(V_1^* \otimes V_2\). This motivates the following.

2.4.4. Proposition. Let \(V_1, V_2\) be two finite-dimensional \(\mathbb{K}\)-vector spaces.

(i) There is a natural \(\mathbb{K}\)-linear isomorphism:

\[
\text{Hom}_\mathbb{K}(V_1, V_2) \simeq V_1^* \otimes_\mathbb{K} V_2 \quad [\mathbb{K}\text{-Mod}].
\]

(ii) If in addition \(V_1, V_2\) are representations of \(G\), then the above is even an isomorphism of representations, viz.:

\[
\text{Hom}_\mathbb{K}(V_1, V_2) \simeq V_1^* \otimes_\mathbb{K} V_2 \quad [\mathbb{K}[G]\text{-Mod}].
\]

In (ii) one has a \(\mathbb{K}[G]\)-isomorphism, but one still tensors over \(\mathbb{K}\).

Proof.

(i) First notice that dimensions agree, so it is enough to find a linear injection. Let \(B = \{e_1, \ldots, e_n\}\) be a basis of \(V_1\) and \(B^* = \{e_1^*, \ldots, e_n^*\}\) be the dual basis.

To \(f \in \text{Hom}_\mathbb{K}(V_1, V_2)\), associate the element:

\[
T(f) = \sum_{i=1}^n e_i^* \otimes f(e_i) \in V_1^* \otimes_\mathbb{K} V_2.
\]

This is well-defined. Moreover, \(T : \text{Hom}_\mathbb{K}(V_1, V_2) \to V_1^* \otimes_\mathbb{K} V_2\) is clearly linear.

Now suppose \(T(f) = 0\). Expressing \(f(e_i)\) over any basis of \(V_2\), we see that the non-zero \(e_i^* \otimes f(e_i)\) are linearly independent in \(V_1^* \otimes V_2\). So all are zero,
meaning \( f(e_i) = 0 \) on \( B \). This implies \( f = 0 \), and \( T \) is injective. The dimensional argument gives the conclusion.

(ii) Let \( f: V_i \to V_j \) and \( g \in G \). We shall compute and compare \( T(g \cdot f) \) with \( g \cdot T(f) \).

Consider the linear map:

\[
\begin{align*}
\varphi & \mapsto \varphi \circ g^{-1}.
\end{align*}
\]

Let \( M = \text{Mat}_G(g^{-1}) \) have coefficients \((m_{i,j})\), so that \( g^{-1}(e_i) = \sum_i m_{i,j} e_j \).

Then by Lemma 2.2.3, we know that \( \text{Mat}_G u = M' \). This means that \( u(e_j^*) = \sum_i m_{j,i} e_i^* \). We are ready for computation.

\[
\begin{align*}
T(g \cdot f) &= T[x \mapsto g f(g^{-1} x)] \\
&= \sum_i e_i^* \otimes gf(m_{k,j}e_k) \\
&= \sum_{i,k} (m_{k,i}e_i^*) \otimes (g \cdot f(e_k)) \\
&= \sum_k u(e_k^*) \otimes (g f(e_k)) \\
&= \sum_k (e_k^* \circ g^{-1}) \otimes (g \cdot f(e_k)) \\
&= \sum_k (g \cdot e_k^*) \otimes (g \cdot f(e_k)) \\
&= g \cdot \left( \sum_k e_k^* \otimes f(e_k) \right) \\
&= g \cdot T(f).
\end{align*}
\]

We are done. \( \square \)

**2.4.5. Remark.** This does not hold if \( \dim V_i \) is infinite; see exercise 2.5.2. It is however enough to have \( \dim V_i < \infty \); see exercise 2.5.3.

### 2.5 Exercises

**2.5.1. Exercise.** Let \( V_i, V_j, V_k \) be \( \mathbb{K} \)-vector spaces.

1. Show that there are natural linear isomorphisms: \( \bullet V_i \oplus V_j \cong V_i \oplus V_j \), \( \bullet V_i \oplus (V_j \oplus V_k) \cong (V_i \oplus V_j) \oplus V_k \), \( \bullet V_i \oplus V_j \cong V_i \oplus V_j \), \( \bullet V_i \oplus V_j \cong (V_i \oplus V_j) \), \( \bullet V_i \oplus V_j \cong V_i \oplus V_j \).

2. Show that if \( V_i, V_j \) are \( \mathbb{K} \)-linear representation of \( G \), the above is an isomorphism of representations.

**2.5.2. Exercise.** Let \( V, W \) be \( \mathbb{K} \)-vector spaces. Show that in general, \( V^* \otimes_{\mathbb{K}} W \cong \{ \varphi \in \text{Hom}_{\mathbb{K}}(V, W) : \dim \text{im} \varphi < \infty \} \).

**2.5.3. Exercise.** Show that Proposition 2.4.4 still holds if \( V_i \) is finite-dimensional, regardless of \( V_j \). Hint: explicitly give the converse isomorphism.
2.5.4. Exercise. Let $V_1, V_2$ be two $\mathbb{K}$-vector spaces and $\beta: V_1 \times V_2 \to \mathbb{K}$ be a bilinear form. Suppose $\beta$ is non-degenerate, viz:

1. Find natural linear embeddings $V_i \to V_i^*$ and $V_j^* \to V_j$.

2. Deduce that if $V_i$ or $V_j$ is finite-dimensional, then both are and $V_i \cong V_j^*$.

2.5.5. Exercise. For this exercise it is preferable to know about quotient vector spaces. Let $V$ be a finite-dimensional vector space.

1. For $W \leq V$ a subspace, let $W^\perp = \{ \varphi \in V^* : (\forall w \in W)(\varphi(w) = 0) \}$. Prove that $\dim W^\perp = \dim V - \dim W$.

2. Suppose in addition that $V$ is a representation of $G$. Prove that $V$ is irreducible iff $V^\perp$ is.

(* 2.5.6. Exercise. Return to Proposition 2.4.4. Prove that $T(f)$ does not depend on the choice of the basis. Deduce another proof that $T$ is $G$-covariant.

3 Around reducibility

Abstract. Schur's Lemma (§ 3.1) describes arrows between irreducible representations; there are extra claims if $\mathbb{K}$ is algebraically closed but the general part is worth remembering. Maschke's Theorem (§ 3.2) provides nice direct sum decompositions and eliminates the need for quotient objects, but has assumptions on char $\mathbb{K}$. We then introduce isotypical components (§ 3.3).

From this section on, it is important to distinguish assumptions on algebraic closedness (‘à la Schur+) from assumptions on the characteristic (‘à la Maschke’).

3.1 Schur’s Lemma

The phrase ‘Schur’s Lemma’ refers to various statements about morphisms between irreducible representations. Some are extremely general; some hold in finite-dimensional spaces; some require, in addition, the base field to be algebraically closed. But there are no assumptions on the characteristic.

3.1.1. Lemma (Schur’s Lemma). Let $G$ be a group and $\mathbb{K}$ be a field. Let $(V_1, \rho_1), (V_2, \rho_2)$ be irreducible $\mathbb{K}$-linear representations.

(i) If $f: \rho_1 \to \rho_2$ is a morphism of representations, then either $f = 0$ or $f$ is an isomorphism.

(ii) In particular, if $(V, \rho)$ is an irreducible representation, then $\text{End}_{\mathbb{K}[G]}(V)$ is a skew-field.
(iii) Suppose that $\mathbb{K}$ is algebraically closed. If $(V, \rho)$ is a finite-dimensional, irreducible representation over $\mathbb{K}$, then $\text{End}_{\mathbb{K}[G]}(V) = \mathbb{K} \text{Id}_V \simeq \mathbb{K}$.

**Proof.**

(i) Suppose $f \neq 0$. Then $W_i = \ker f < V$. However, $W_i$ is $G$-invariant since for $x \in W_i$ and $g \in G$ one has $f(gx) = g f(x) = g \cdot o = o$. By irreducibility of $V$, one has $W_i = \{ o \}$ and $f$ is injective. Similarly, $W_2 = \text{im } f > o$ is $G$-invariant, hence by irreducibility $W_2 = V$ and $f$ is surjective. It is thus a $G$-covariant linear isomorphism, hence an isomorphism of representations.

(ii) A special case. Recall that the inverse of a $\mathbb{K}$-linear, $G$-covariant isomorphism, is again $\mathbb{K}$-linear and $G$-covariant.

(iii) Of course every scalar map $\lambda \text{Id}_V$ is $G$-covariant, so $\mathbb{K} \text{Id}_V \subseteq \text{End}_{\mathbb{K}[G]}(V)$. We prove the converse. Let $\sigma \in \text{End}_{\mathbb{K}[G]}(V)$. By assumption, $V$ is finite-dimensional over $\mathbb{K}$ and $\mathbb{K}$ is algebraically closed. So any linear endomorphism of $V$ has an eigenvalue. Say $\lambda \in \mathbb{K}$ has a non-zero eigenspace $E_\lambda(\sigma) \neq \{ o \}$. Then $\tau = \sigma - \lambda \text{Id}_V \in \text{End}_{\mathbb{K}[G]}(V)$ has a non-zero kernel. But $\text{End}_{\mathbb{K}[G]}(V)$ is a skew-field by (ii), so $\sigma = \lambda \text{Id}_V$. Hence $\text{End}_{\mathbb{K}[G]}(V) = \mathbb{K} \text{Id}_V \simeq \mathbb{K}$. 

A useful consequence is that over an algebraically closed field, if both $\rho_1$ and $\rho_2$ are irreducible, then $\dim \text{Hom}_{\mathbb{K}[G]}(\rho_1, \rho_2)$ is 0 or 1.

### 3.1.2. Remarks.

- Even starting with a commutative $\mathbb{K}$, one can construct irreducible representations where $\text{End}_{\mathbb{K}[G]}(V)$ is a *non-commutative* skew-field. We return to the topic in § 11; meanwhile see exercise 3.4.1.
- The argument makes crucial use of commutativity of $\mathbb{K}$. For vector spaces over skew-fields, the maps $\lambda \text{Id}_V$ are no longer $\mathbb{K}$-linear. I am however not aware of a developed representation theory over skew-fields.

### 3.2 Maschke’s Theorem

The phrase ‘Maschke’s Theorem’ refers to various statements about expressing arbitrary representations as direct sums of irreducible ones. All require the characteristic to be coprime to the order of the finite group, hence always work in characteristic 0. (Generalisations to infinite groups would require higher-level structure.) But there are no assumptions on algebraic closedness.

#### 3.2.1. Theorem (Maschke’s Theorem). Let $G$ be a finite group and $\mathbb{K}$ be a field. Suppose that $\mathbb{K}$ has coprime characteristic. Let $(V, \rho)$ be a $\mathbb{K}$-linear representation. Then:

(i) Every $G$-invariant subspace $W \leq V$ admits a $G$-invariant direct complement.

(ii) $V$ is a direct sum of irreducible representations.

(iii) In particular, if $V$ is finite-dimensional, then it is a direct sum of finitely many irreducible representations.

#### 3.2.2. Remarks.
• (i) expresses complete reducibility, also known as semisimplicity of the group algebra $\mathbb{K}[G]$. (See §13.2.)

• While (ii) does not require finite-dimensionality (it does require finiteness of $G$ though), it is the kind of claim that often disturbs beginners.

Proof.

(i) Let $W \leq V$ be as in the statement. Because direct complements exist in the category of vector spaces, we may take a linear subspace $S \leq V$ such that $V = W \oplus S$. The problem is that $S$ need not be $G$-invariant. We shall ‘average it’ as follows.

Let $\pi: V \to W$ be the linear projector onto $W$ parallel to $S$. Now let:

$$\hat{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \circ \pi \circ \rho(g) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi g.$$

(The second formula is in implicit notation, which we now use.) We claim the following.

• $\hat{\pi}$ is a linear, $G$-covariant map. Linearity is obvious since $\hat{\pi}$ is a sum of linear maps. Now for fixed $h \in G$, the map $g \mapsto gh$ is a bijection of the indexing set $G$ so:

$$\hat{\pi}h = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi g \cdot h = \frac{1}{|G|} \sum_{g \in G} h \cdot h^{-1} g^{-1} \cdot g \cdot h = \frac{1}{|G|} \sum_{g' \in G} h \cdot g' \pi g' = h \hat{\pi}.$$

• $\hat{\pi}$ is a linear projector onto $W$. Recall that $W$ is $G$-invariant. So for any $g \in G$ and $v \in V$, one has $g v \in V$, then $\pi(g v) \in W$, and $g^{-1} \pi(g v) \in W$. Hence $\hat{\pi}(g) \in W$ and $\text{im} \hat{\pi} \leq W$. Now $\pi$ is the identity on $W$. So for $w \in W$, one has $g w \in W$, then $\pi(g w) = g w$, and $g^{-1} \pi(g w) = w$. Hence $\hat{\pi}(w) = w$.

Summing up, $\text{im} \hat{\pi} \leq W$ and $\hat{\pi}|_W = \text{Id}_W$. This is the definition of a linear projector with image $W$.

Let $\hat{\mathcal{S}} = \ker \hat{\pi}$. Being the kernel of the projector $\hat{\pi}$, it is a direct complement of $W = \text{im} \hat{\pi}$. Moreover it is $G$-invariant, because $\hat{\pi}$ is $G$-covariant. We are done.

(ii) This requires some maximality argument à la Zorn (and the proof may be omitted by the unexperienced).

Let $\mathcal{F}$ be a family of irreducible subrepresentations of $V$ whose sum is direct, and maximal as such. (This exists by Zorn’s lemma, or the axiom of choice and ordinal induction.) Let $S = \sum_{W \in \mathcal{F}} W = \bigoplus_{W \in \mathcal{F}} W$.

If $S < V$, then by (i), there is a $G$-invariant complement $T$. Now $T$ contains an
irreducible representation $W'$ by Lemma 1.2.3. So $\mathcal{F} = \mathcal{F} \cup \{W'\}$ is a family of irreducible representations whose sum is direct. By maximality of $\mathcal{F}$ as such, one has $W' \in \mathcal{F}$. So $W' \leq S$, a contradiction. This proves $S = V$, as wanted.

(iii) A direct consequence of (ii), but it can also be proved directly by induction on $\dim V$. See exercise 3.4.4.

3.2.3. Remarks.

- If $\text{char } \mathbb{K}$ divides $|G|$, then Maschke’s theorem no longer holds (exercise 3.4.6), so the theory of modular representations is more complicated.

- Maschke’s principle fails for infinite groups, even over $\mathbb{C}$. However, it can be salvaged for certain infinite groups bearing extra structure (typically, a Haar measure as used in Lie theory).

3.3 Isotypical components

We proceed to analysing reducible representations in terms of ‘atoms’ (viz. irreducible representations).

3.3.1. Definition. Let $G$ be a group and $\mathbb{K}$ be a field. Let $V$ be a $\mathbb{K}$-linear representation of $G$. Also let $T \in \text{Irr}_{\mathbb{K}}(G)$.

- Let $\text{Cp}_V(T) = \{W \leq V : W$ is a subrepresentation and $W \cong T\}$ be the set of isomorphic copies of $T$ inside $V$.

- Let $\text{Iso}_V(T) = \sum_{W \in \text{Cp}_V(T)} W$ be their sum, called the isotypical component of $V$ of type $T$.

These notions behave extremely well in presence of both Schur’s and Maschke’s phenomena. The following can be read following the analogy with $p$-primary components of abelian groups.

3.3.2. Theorem. Let $G$ be a finite group and $\mathbb{K}$ be a field of coprime characteristic. Let $V$ be a $\mathbb{K}$-linear representation. Then:

(i) Every irreducible subrepresentation of $\text{Iso}_V(T)$ is isomorphic to $T$.

(ii) $V = \bigoplus_{T \in \text{Irr}_{\mathbb{K}}(G)} \text{Iso}_V(T)$.

(iii) If $V_1, V_2$ are two $\mathbb{K}$-linear representations, and $f : V_1 \to V_2$ is a morphism of representations, then $f(\text{Iso}_{V_1}(T)) \leq \text{Iso}_{V_2}(T)$.

(iv) For every $T \in \text{Irr}_{\mathbb{K}}(G)$, there is a subfamily $\mathcal{F}_T \subseteq \text{Cp}_V(T)$ such that $\text{Iso}_V(T) = \bigoplus_{W \in \mathcal{F}_T} W$.

3.3.3. Remarks.

- Subrepresentations $\text{Iso}_V(T)$ are completely canonical, which is confirmed by (iii).

- However a family $\mathcal{F}_T$ as in (iv) is highly non-canonical. The simplest example is a 2-dimensional vector space $V$ as a representation of $\{1\}$. Certainly $V$ is a direct sum of two vector lines, but these are not uniquely determined.
The theorem does not require algebraic closedness, but fails badly in characteristic dividing \(|G|\).

**Proof.** We use quick lemmas.

**3.3.4. Lemma.** Suppose \( V', W \leq V \) are subrepresentations with \( W \) irreducible. Then \( W \cap V' = \{0\} \) or \( W \leq V' \).

**Proof.** Let \( R = W \cap V' \), which is a subrepresentations of \( W \). If \( R = \{0\} \) we are done. Otherwise, by irreducibility, \( W = R \leq V' \).

**3.3.5. Lemma.** Every sum \( \sum_i W_i \) of irreducible subrepresentations \( W_i \leq V \) is the direct sum of a subfamily \( f \in I \).

**Proof.** Let \( S = \sum_i W_i \) be a sum of irreducible subrepresentations. Let \( J \subseteq I \) be a subfamily whose sum is direct, and maximal with respect to this property. (This exists by maximality principles à la Zorn.) Now let \( S' = \sum_i W_j = \bigoplus J W_j \). We claim that \( S = S' \).

If it is not the case, there is \( i \in I \) with \( W_i \notin S' \). By Lemma 3.3.4, \( W_i \cap S' = \{0\} \). So \( J \cup \{i\} \) is a family properly containing \( J \), whose sum is direct: a contradiction. So \( S = S' = \bigoplus J W_j \) is the direct sum of a subfamily.

**3.3.6. Lemma.** Suppose that all \( W_i \) for \( i \in I \) and \( W \) are irreducible subrepresentations of \( V \), with \( W \leq \sum_i W_i \). Then \( W \) is isomorphic to one of the \( W_i \).

**Proof.** By Lemma 3.3.5, up to taking a subfamily we may assume \( \sum_i W_i = \bigoplus J W_i \).

So we may consider the projectors \( \pi_i \) onto \( W_i \) parallel to the other summands. Since \( W \neq \{0\} \), there is \( i \in I \) such that \( \pi_i(W) \neq 0 \). We fix one such and let \( f \) be the restriction \( \pi_i|_W : W \to W_i \). Then \( f \neq 0 \). Now both \( W \) and \( W_i \) are irreducible, so by Schur’s Lemma \( f \) is an isomorphism \( W \cong W_i \).

(i) Immediate from Lemma 3.3.6.

(ii) By Maschke’s theorem, \( V \) is a sum of irreducible representations. Each lies in some \( \text{Iso}_V(T_i) \), whence \( V = \sum_{T \in \text{Irr}(G)} \).

We prove that the latter sum is direct; suppose not. Then there are distinct types \( T_0, T_1, \ldots, T_n \) with \( n \geq 1 \) such that \( \text{Iso}_V(T_0) \cap (\sum_{i=1}^n \text{Iso}_V(T_i)) \neq \{0\} \). So there is \( W' \in \text{Cp}_V(T_0) \) contained in \( \sum_{i=1}^n \text{Iso}_V(T_i) \). By Lemma 3.3.6, \( T_0 \) is isomorphic to one summand of some \( \text{Iso}_V(T_i) \), viz. \( T_0 = T_i \), a contradiction.

(iii) Let \( W_i \in \text{Cp}_V(T) \). Then \( f(W_i) \leq V_i \) is \( G \)-invariant, hence a subrepresentation.

By Schur’s Lemma, either \( f(W_i) = \{0\} \) or \( f(W_i) \cong W_i \cong T \). So in either case, \( f(W_i) \leq \text{Iso}_V(T) \).

(iv) Immediate from Lemma 3.3.5.

**3.3.7. Corollary** (and definition). Let \( G \) be a finite group and \( \mathbb{K} \) be a field of coprime characteristic. Let \( V \) be a \( \mathbb{K} \)-linear, finite-dimensional representation. Then there are well-
defined integers \( n_T = n_T(V) \) for \( T \in \text{Irr}_K(G) \) such that:

\[
V \cong \bigoplus_{T \in \text{Irr}_K(G)} T^{n_T} [K[G]-\text{Mod}].
\]

The integer \( n_T(V) \) is called the multiplicity of \( T \) in \( V \).

**Proof.** The existence is a reformulation of Theorem 3.2.1 (iii). Write \( V \) as a direct sum of irreducible representations; now sort them according to their isomorphism types.

It remains to show that the integers do not depend on the decomposition. Suppose there is an isomorphism of representations \( f: \bigoplus T^{n_T} \cong \bigoplus T^{m_T} \). By Theorem 3.2.2 (iii), it restricts to isomorphisms \( f_T: T^{n_T} \cong T^{m_T} \) for each \( T \in \text{Irr}_K(G) \). But then, \( n_T \dim T = m_T \dim T \), so \( n_T = m_T \), as wanted.

Hence a finite-dimensional, \( K \)-linear representation of a finite group over a field of coprime characteristic is entirely determined by the number of ‘atoms’ (viz. irreducible representations) of each type in it.

3.3.8. **Remark.** Corollary 3.3.7 can be used to give a proof in coprime characteristic that a representation \( V \) is irreducible if and only if \( V^* \) is. (This holds with no assumptions on the characteristic: see exercise 2.5.5.)

Suppose \( V \) is irreducible and write \( V^* \cong \bigoplus T^{n_T} [K[G]-\text{Mod}] \). Then \( V \cong V^{**} \cong \bigoplus (T^*)^{n_T} [K[G]-\text{Mod}] \). But \( V \) is irreducible, so there is only one term, and it has multiplicity 1. Hence \( V^* \) is irreducible. The converse also uses \( V^{**} \cong V [K[G]-\text{Mod}] \).

In § 5, exercise 5.6.6 will give an explicit formula for the projector \( \pi_T \) onto \( \text{Iso}_V(T) \) parallel to the sum of the other isotypical components.

### 3.4 Exercises

3.4.1. **Exercise.** Let \( \mathbb{H}_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \) be the group of basic quaternions, satisfying:

\[
i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.
\]

Let \( V \cong \mathbb{R}^4 \) have basis \( 1, i, j, k \), and extend linearly to define an \( \mathbb{R} \)-representation of \( \mathbb{H}_8 \) in \( V \). Prove that it is irreducible, and determine \( \text{End}_{\mathbb{R}[G]}(V) \).

3.4.2. **Exercise.** Let \( G \) be a finite group and \( \rho \in \text{Irr}_C(G) \). Show that if \( \rho \) is injective, then \( Z(G) \) is cyclic.

3.4.3. **Exercise.** Let \( \rho \) be the permutation representation of \( G = \text{Sym}(3) \), viz. the permutation action on \( V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \). Check that \( L = \{ e_1 + e_2 + e_3 \} \) is \( G \)-invariant. Give a \( G \)-invariant complement. (Exercise 3.4.5 gives an instant method.)

3.4.4. **Exercise.** Return to Theorem 3.2.1. Suppose \( \text{char } K = 0 \) and \( \dim V < \infty \). Prove (iii) by induction on \( \dim V \) without using (ii).

3.4.5. **Exercise** (an alternative proof of Maschke’s theorem over \( \mathbb{C} \)). Let \( G \) be a group and \( V \) be a finite-dimensional, complex, linear representation of \( G \). Let \( \langle \cdot, \cdot \rangle \) be a complex scalar product, viz. a sesquilinear, Hermite-symmetric, positive definite form \( V \times V \to \mathbb{C} \).
1. Let \( [x|y] = \frac{1}{|G|} \sum_{g \in G} [gx|gy] \). Prove that this is a complex scalar product.

2. Prove that for \( x, y \in V \) and \( g \in G \) one has \( [gx|gy] = [x|y] \).

3. Deduce a proof of Maschke's theorem over \( \mathbb{C} \).

4. Extra question: if \( V \) is irreducible, show that all \( G \)-invariant complex scalar products are multiples of \( [|] \).

**Exercise** (failure of Maschke in native characteristic). Let \( G = C_2 \) be the cyclic group with two elements. Let it act on \( V = \mathbb{F}_2^3 \) by taking the generator to:

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

Prove that \( \mathbb{F}_2 e_2 \) is \( G \)-invariant, but has no \( G \)-invariant complement.

**Exercise.** Let \( G \) be finite. Show that if all irreducible, linear complex representations are 1-dimensional, then \( G \) is abelian.

**Exercise.** Determine which lemmas and which ads of Theorem 3.3.2 remain true

- if \( \text{char } \mathbb{K} \) divides \( |G| \),
- if \( G \) is infinite.

4 **Characters**

**Abstract.** The definition of a character (§ 4.1) first looks 'too simple to be useful,' and yet is extremely powerful. Character tables encode the values of the irreducible characters. Characters are typical examples of class functions (§ 4.2). We then describe characters of representations obtained by the usual algebraic constructions (§ 4.3).

**4.1 Characters and character tables**

**Definition.** Let \( G \) be a group and \( \mathbb{K} \) be a field. Let \( \rho : G \to \text{GL}(V) \) be a finite-dimensional, \( \mathbb{K} \)-linear representation. Its **character** is the map

\[
\chi_\rho : G \to \mathbb{K},
\]

\[
g \mapsto \text{tr } \rho(g),
\]

where \( \text{tr} \) denotes the trace. (When there is no ambiguity on \( \rho \), one simply writes \( \chi \).)

**Remarks.**

- A character need not be a morphism!

- Since the trace is invariant under conjugation, so is any character, viz. one has \( \chi(g^{-1}hg) = \chi(h) \). In particular, one really computes \( \chi(\gamma) \) for \( \gamma \) a conjugacy class of \( G \).

- Characters are especially useful if \( \mathbb{K} \) is algebraically closed (so we have eigenvalues) and has coprime characteristic \( o \) (so we have complete reducibility). One cannot imagine at first the strength of character theory over \( \mathbb{C} \).
• It follows immediately from Corollary 3.3.7 that in coprime characteristic, for every character \( \chi_V \) there are integers \( n_T \) for \( T \in \text{Irr}_K(G) \) such that:
\[
\chi_V = \sum_{T \in \text{Irr}_K(G)} n_T \chi_T.
\]

• Interestingly enough, the \( n_T \)'s above are integers of \( k \), meaning that in positive characteristic \( p \) they are to be considered modulo \( p \). Thus characters can, at best, detect multiplicity modulo the characteristic.

For instance, the character of \( \text{triv} \oplus T \) is \( \chi_{\text{triv}} + p \chi_T = \chi_{\text{triv}} \). But as representations, \( \text{triv} \neq (\text{triv} \oplus T) \). So parts of character theory even require char \( k = 0 \).

4.1.3. Lemma. Suppose \( k \leq C \). Let \( G \) be a finite group and \( \rho: G \to \text{GL}(V) \) be a \( k \)-linear representation of \( G \) with character \( \chi \). Then for all \( g \in G \):
\[
\chi(g^{-1}) = \overline{\chi(g)}.
\]

\[\text{Proof.}\] We know that \( \rho(g) \) can be brought to diagonal form, say one of its matrices is \( \text{diag}(\lambda_1, \ldots, \lambda_n) \). Hence \( \rho(g^{-1}) = \rho(g)^{-1} \) is diagonalisable to \( \text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) \). But each \( \lambda_i \) is a complex root of unity, so \( \lambda_i^{-1} = \overline{\lambda_i} \). Therefore one of the matrices of \( g^{-1} \) is \( \text{diag}(\overline{\lambda_1}, \ldots, \overline{\lambda_n}) \), with trace:
\[
\chi(g^{-1}) = \text{tr} \rho(g^{-1}) = \text{tr} \text{diag}(\overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \sum_i \overline{\lambda_i} = \sum_i \lambda_i = \overline{\text{tr} \rho(g)} = \overline{\chi(g)}. \quad \square
\]

4.1.4. Definition.

• An irreducible character is the character of an irreducible representation.

We let \( \text{Irr}_K(G) \) be the set of irreducible characters. Because characters will determine representations (at least in the irreducible case over good fields), this does not conflict with the same notation for the set of irreducible representations.

• The character table of a finite group \( G \) over \( k \) is the table constructed as follows:

  - list all conjugacy classes \( y_1, \ldots, y_r \) of \( G \), with respective number of elements say \( d_1, \ldots, d_r \).
  - list all irreducible characters \( \chi_1, \ldots, \chi_r \) of \( G \) over \( k \). For good \( k \), it is the same number \( r \) (Theorem 5.1.1).
  - Now tabulate values as follows:

\[
\begin{array}{c|ccc}
G & y_1 \ [x_1 d_1] & \cdots & y_r \ [x_r d_r] \\
\chi_1 & \chi_1(y_1) & \cdots & \chi_1(y_r) \\
\vdots & \vdots & & \vdots \\
\chi_r & \chi_r(y_1) & \cdots & \chi_r(y_r) \\
\end{array}
\]

When not otherwise specified, character tables are usually given over \( C \) (or \( \mathbb{C} \)).

4.1.5. Example. Here is the character table of \( \text{Sym}(3) \).

\[
\begin{array}{c|ccc}
 & 1 & (12) & (132) & (213) \\
\chi_{\text{triv}} & 1 & 1 & 1 & 1 \\
\chi_1 & 1 & -1 & 1 & 1 \\
\chi_2 & 2 & 0 & -1 & 1 \\
\end{array}
\]

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4.1.6. Remarks.

- Such a table does not really give the representations themselves, nor the group structure.
  For instance, the dihedral group $D_{2\cdot4}$ and the basic quaternion group $H_{8}$ have the same character table over $\mathbb{C}$, although they are non-isomorphic.
- This cannot happen with finite simple groups, but we know this because we have the full list of them. Actually the finite simple groups are determined by much less information than their character tables—again because we have the list.
- However character tables encode much information on finite groups. Actually some properties can be ’read off’ character tables; see exercise 7.4.2.

4.2 Class functions

Invariance under conjugation begs for a definition.

4.2.1. Definition. A class function (also: central function) is a function $\alpha : G \rightarrow \mathbb{K}$ satisfying: $(\forall g)(\forall h)(\alpha(g^{-1}hg) = \alpha(g))$.

More algebraically, these are functions which factor through the conjugation relation. For a class function $\alpha$ and a conjugacy class $\gamma$, it makes sense to write $\alpha(\gamma)$.

4.2.2. Lemma (and notation). Let $G$ be a finite group and $\mathbb{K}$ be a field of coprime characteristic.

(i) Class functions form a $\mathbb{K}$-vector subspace of the space $\mathbb{K}^G$ of all maps $G \rightarrow \mathbb{K}$.

We denote it by $C_\mathbb{K}(G)$ (or simply $\mathcal{C}$ if there is no ambiguity on $\mathbb{K}$ or $G$).

(ii) dim$_\mathbb{K} C_\mathbb{K}(G)$ is the number of conjugacy classes of $G$.

(iii) $C_\mathbb{K}$ bears a bilinear, symmetric, non-degenerate form given by:

$$ (\alpha | \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g^{-1})\beta(g). $$

(iv) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{C}_\mathbb{C}$ also bears a complex scalar product, given by:

$$ [\alpha | \beta] = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\beta(g). $$

(v) When $\mathbb{K} \subseteq \mathbb{C}$ and $\chi_1, \chi_2$ are characters, both coincide: $(\chi_1 | \chi_2) = [\chi_1 | \chi_2]$.

Actually (i) and (ii) do not require coprimality of the characteristic (as seen from the proof). But dividing by $|G|$ certainly does.

Proof.

(i) is clear.

(ii) For $\gamma \subseteq G$ a conjugacy class, let $1_{\gamma}$ be the indicator function, viz. the function which is 1 on $\gamma$ and 0 elsewhere. This map is in $\mathcal{C}$. Let $B = \{1_{\gamma} : \gamma \text{ a conjugacy class}\}$. We claim that it is a basis of $C_\mathbb{K}(G)$. Indeed, suppose $\sum \lambda_{\gamma}1_{\gamma} = 0$ in
obvious notation. Let $\delta$ be a conjugacy class and let $g \in \delta$; then $\sum \lambda_\gamma \tau_\gamma(g) = \lambda_\delta = 0$, which applies to every class. Now if $\alpha \in C$, then for every class $\gamma$ let $\lambda_\gamma = \alpha(\gamma)$. Then $\alpha$ and $\sum \lambda_\gamma \tau_\gamma$ agree everywhere.

(iii) Bilinearity is clear. Symmetry follows from reindexing:

$$(\beta | \alpha) = \frac{1}{|G|} \sum_{g \in G} \beta(g^{-1}) \alpha(g) = \frac{1}{|G|} \sum_{g' \in G} \beta(g') \alpha(g'^{-1}) = (\alpha | \beta).$$

We prove non-degeneracy. Let $\alpha \in C_K$ be $(\cdot | \cdot)$-orthogonal to all class functions. Let $\gamma$ be a conjugacy class; then so is $\gamma^{-1} = \{ g^{-1} : g \in \gamma \}$. Let $f = 1_{\gamma^{-1}}$ be the indicator function of $\gamma^{-1}$. Then since $\alpha$ is a class function:

$$0 = |G| \cdot (f \alpha) = \sum_{g \in G} f(g^{-1}) \alpha(g) = \sum_{g' \in G} 1_{\gamma^{-1}}(g'^{-1}) \alpha(g) = \# \gamma \cdot \alpha(\gamma).$$

Now $\# \gamma = [G : C_G(g)]$ for any $g \in \gamma$, and therefore $\# \gamma$ divides $|G|$. In particular, $\# \gamma$ is coprime to the characteristic (if positive), and it follows $\alpha(\gamma) = 0$. This holds for any conjugacy class: so $\alpha$ is zero globally.

(iv) Actually the formula defines a complex vector space on all of $C^G$, as easily seen. Since $C_C$ is a vector subspace, the claim follows.

(v) Clear since by Lemma 4.1.3, $\chi(g^{-1}) = \overline{\chi(g)}$ for a character. □

4.2.3. Remarks.

- There are other conventions (inverse on the right, complex-conjugation on the right). What matters is the resulting theory, viz. side-invariant phenomena.
- Over $\mathbb{C}$ there are two natural pairings of class functions: $(\cdot | \cdot)$ and $[\cdot | \cdot]$. They return the same values on characters but need not agree on all class functions. Only the former is symmetric; the latter is merely Hermite-symmetric.
- In particular they give rise to distinct notions of orthonormality, which however agree on characters.
- Analysts will favour working with $[\cdot | \cdot]$, and algebraists will prefer $(\cdot | \cdot)$.

4.3 Characters and algebraic constructions

§ 2 gave methods how to obtain new representations from existing ones. The proposition below will describe the consequent behaviour of characters. It builds on the principle that isomorphic representations have equal character.

4.3.1. Lemma. Let $G$ be a finite group and $\mathbb{K}$ be a field. If $\rho_1 \simeq \rho_2$ are isomorphic finite-dimensional representations, then $\chi_1 = \chi_2$.

**Proof.** By definition, there is an isomorphism $f: \rho_1 \simeq \rho_2$, viz. a linear isomorphism
\( V_i \cong V_2 \) such that for all \( g \in G \):

\[
\begin{array}{ccc}
V_i & \overset{f}{\longrightarrow} & V_2 \\
\rho_i(g) & \downarrow & \rho_i(g) \\
V_i & \overset{f}{\longrightarrow} & V_2
\end{array}
\]

Another way to write it is \( \rho_2(g) = f^{-1} \rho_i(g) f \). Taking matrices if necessary, one sees \( \text{tr} \rho_2(g) = \text{tr} \rho_i(g) \) for all \( g \in G \), that is, \( \chi_1 = \chi_2 \).

\[ \square \]

**4.3.2. Remark.** A converse will be seen: in good cases, representations with equal characters are actually isomorphic (Theorem 5.1.1).

We return to the constructions of §2 and determine their characters.

**4.3.3. Proposition.** Let \( G \) be a group and \( \mathbb{K} \) be a field. Let \((V, \rho), (V_i, \rho_i), (V_2, \rho_2)\) be \( \mathbb{K} \)-linear, finite-dimensional representations with characters \( \chi, \chi_1, \chi_2 \).

(i) The character of \( \rho_1 \otimes \rho_2 \) is \( \chi_1 + \chi_2 \).

(ii) The character of \( \rho^* \) is \( \chi^*(g) = \chi(g^{-1}) \). In case \( \mathbb{K} \subseteq \mathbb{C} \), this also equals \( \overline{\chi(g)} \).

(iii) The character of \( \rho_1 \otimes \mathbb{K} \rho_2 \) is \( \chi_1 \chi_2 \).

(iv) The character of Hom\( \mathbb{K}(\rho_1, \rho_2) \) is \( \chi_1^* \chi_2 \). In case \( \mathbb{K} \subseteq \mathbb{C} \), this also equals \( \overline{\chi_1} \cdot \chi_2 \).

**Proof.**

(i) By definition, \( \rho_1 \otimes \rho_2 \) is the natural action of \( G \) on \( V_1 \otimes V_2 \), viz. \( G \) acts on \( V_1 \) via \( \rho_1 \) and on \( V_2 \) via \( \rho_2 \). So:

\[
\chi_{\rho_1 \otimes \rho_2}(g) = \text{tr}(\rho_1 \otimes \rho_2)(g) = \text{tr} \rho_1(g) + \text{tr} \rho_2(g) = \chi_1(g) + \chi_2(g).
\]

(ii) By definition, \( \rho^*(g) : V^* \rightarrow V^* \) takes a linear form \( \varphi \) to the linear form \( v \rightarrow \varphi(g^{-1} \cdot v) \). Working in coordinates if necessary, if \( B \) is a basis of \( V \) then:

\[
\text{Mat}_B^* \rho^*(g) = (\text{Mat}_B \rho(g^{-1}))^t,
\]

so \( \chi^*(g) = \chi(g^{-1}) \). In case \( \mathbb{K} \subseteq \mathbb{C} \), this also equals \( \overline{\chi(g)} \) by Lemma 4.1.3.

(iii) By definition, \( \rho_1 \otimes \rho_2 \) is the tensor product of \( G \) on \( V_1 \otimes V_2 \), viz. \( G \) acts on \( v_1 \otimes v_2 \) by \( (\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2) \). Working in coordinates if necessary,

\[
\chi_{\rho_1 \otimes \rho_2}(g) = \text{tr}(\rho_1 \otimes \rho_2)(g) = \text{tr} \rho_1(g) \cdot \text{tr} \rho_2(g) = (\rho_1 \rho_2)(g).
\]

(iv) Recall from Proposition 2.4.4 (ii) that there is an isomorphism of representations (viz. a \( \mathbb{K}[G] \)-isomorphism):

\[
\text{Hom}_{\mathbb{K}[G]}(V_1, V_2) \cong V_1^* \otimes_{\mathbb{K}} V_2 \quad [\mathbb{K}[G]\cdot\text{-Mod}].
\]

By Lemma 4.3.1 it is enough to give the character of the right-hand, and the claim follows from (ii) and (iii).

\[ \square \]
4.3.4. Remark. Thus the character of $T^n$ is $n\chi_T$. If $K$ has positive characteristic dividing $n$, this is the zero map. Similarly, $\text{triv} \otimes T^n$ has character $\chi_{\text{triv}}$. This simply tells us that the expected function:

$$\{\text{finite-dimensional } K\text{-linear representations}\} \rightarrow \{\text{characters}\}$$

cannot be injective. This is as deep as saying that in positive characteristic, the base field does not have infinitely many integers.

4.4 Exercises

4.4.1. Exercise.

1. Compute $\chi_{\text{reg}}$ for any finite group $G$.
2. Generalise to $\chi_{\text{perm}}$: prove that $\chi_{\text{perm}}(g) = \#\{x \in X : gx = x\}$.
3. Let $G = \text{Sym}(3)$. Inside perm, consider the line $L = \langle e_1 + \cdots + e_3 \rangle$. Let $V$ be a $G$-invariant complement of $L$ inside perm. Give $\chi_V$.
4. Same question when $n = 4$.

4.4.2. Exercise. Give the character table of the cyclic group $C_n$ over $\mathbb{C}$.

For the next exercises, admit that the number of complex, irreducible representations of a finite group equals the number of conjugacy classes (Theorem 5.1.1).

4.4.3. Exercise. Give the character table of $\text{Sym}(3)$ over $\mathbb{C}$.

4.4.4. Exercise. Give the character table of $\text{Alt}(4)$ over $\mathbb{C}$. Hint: act on a regular tetrahedron in the usual 3-dimensional space.

5 Orthogonality relations

Abstract. The main Theorem (§ 5.1) says that irreducible complex characters of a finite group form an orthonormal basis of the space of class functions. There are numerous consequences, such as: every finite-dimensional, complex representation is determined by its character, or: every irreducible, complex representation occurs in the regular representation with multiplicity equal to its dimension. The proof builds on a simple lemma giving the dimension of the subspace of fixed points of a representation (§ 5.2). Orthonormality is then proved in § 5.3 and generation in § 5.4.

5.1 The main theorem

We fix a finite group $G$ and a good field $K$. Recall from § 4.2 that a $K$-valued class function on a group $G$ is a map $\alpha : G \rightarrow K$ such that $(\forall x)(\forall y)(\alpha(xy) = \alpha(x))$.

- Class functions form a $K$-vector space $C_K(G)$, and $\dim_K C_K(G) = \#\text{Conj}(G)$, the number of conjugacy classes of $G$.
- Characters are class functions.
• $C_K(G)$ bears a natural bilinear, symmetric, non-degenerate form:

$$(a|\beta) = \frac{1}{|G|} \sum_{g \in G} a(g^{-1})\beta(g).$$

• If $\mathbb{K} = \mathbb{C}$, then $C_C(G)$ also bears a natural complex scalar product:

$$[a|\beta] = \frac{1}{|G|} \sum_{g \in G} \overline{a(g)}\beta(g).$$

• These agree on characters, so for that matter one may work with either.

5.1.1. **Theorem.** Let $G$ be a finite group and $\mathbb{K}$ be a good field.

(i) $\text{Irr}_K(G)$ forms an orthonormal basis of the space of class functions on $G$. In particular, $\#\text{Irr}_K(G) = \dim_K C_K(G) = \#\text{Conj}(G)$.

In case $\mathbb{K} \subseteq \mathbb{C}$, the same holds with respect to $[\cdot] \cdot$-orthonormality.

(ii) Let $V$ be a $\mathbb{K}$-linear, finite-dimensional representation. Then, as functions from $G$ to $\mathbb{K}$, one has $\chi_V = \sum_{\chi \in \text{Irr}_K(G)} (\chi_V|\chi_T) \chi_T$.

(iii) Every irreducible representation is determined by its character. If $\text{char} \mathbb{K} = 0$, then every finite-dimensional representation is determined by its character.

(iv) Let $\text{reg}$ be the regular representation of $G$ over $\mathbb{K}$. Then $\text{reg} \simeq \bigoplus_{\chi \in \text{Irr}_K(G)} \chi \dim T$.

5.1.2. **Remarks.**

• As a consequence of (iii), it is safe to let $\text{Irr}_K(G)$ be the set of irreducible characters. Likewise, it is safe to write $\text{Iso}_V(\chi)$ instead of $\text{Iso}_V(T)$.

• The general case in (iii) fails in characteristic $p > 0$ (even over good fields). Indeed, for any representation $W$, letting $V = \text{triv} \oplus W^p$ one gets $\chi_V = \text{triv}$. But $V \not\simeq \text{triv} \ [\mathbb{K}[G]\cdot\text{Mod}]$.

• Another way to write (iv) is:

$$\chi_{\text{reg}} = \sum_{\chi \in \text{Irr}_K(G)} \chi(1) \cdot \chi.$$

This immediately implies:

$$|G| = \dim \text{reg} = \chi_{\text{reg}}(1) = \sum_{\chi \in \text{Irr}_K(G)} \chi(1)^2,$$

so $|G|$ is a sum of $\#\text{Conj}(G)$-many squares.

5.1.3. **Remark.** Although $\text{Conj}(G)$ and $\text{Irr}_K(G)$ have the same number of elements (so they are equipotent), there is in general no distinguished bijection between them.

An interesting exception is the symmetric group $\text{Sym}(n)$ where one can attach to each conjugacy class an irreducible character, in a systematic way. This is the theory of Young tableaux.\(^4\)

\(^4\)For example, see § 28 in the Curtis-Reiner book (‘Further reading’).
5.2 Trivial spaces

5.2.1. Definition. Let $V$ be a $\mathbb{K}$-linear representation of a group $G$. The space of fixed vectors, or the $G$-trivial subspace of $V$, is the subspace:

$$C_V(G) = \{ v \in V : (\forall g \in G)(g v = v) \}.$$ 

It should be obvious that $C_V(G)$ is indeed a linear subspace. An alternative notation is $V^G$; we avoid it.

5.2.2. Lemma. Let $G$ be a finite group and $\mathbb{K}$ be a field of coprime characteristic. Let $V, V_1, V_2$ be $\mathbb{K}$-linear, finite-dimensional representations with character $\chi, \chi_1, \chi_2$.

(i) $\dim_{\mathbb{K}} C_V(G) = \frac{1}{|G|} \sum_{g \in G} \chi(g)$.

(ii) $\dim \text{Hom}_{\mathbb{K}(G)}(V, V_2) = (\chi_1, \chi_2)$.

One should be careful that these are formulas in $\mathbb{K}$. The right-hand could be $0$ for a bad reason; typically if the left-hand is divisible by the characteristic.

Proof.

(i) Consider the following linear endomorphism of $V$:

$$\pi = \frac{1}{|G|} \sum g.$$ 

For $h \in G$ one has:

$$h \circ \pi = \frac{1}{|G|} \sum h g = \frac{1}{|G|} \sum^{h \pi}_{g \in G} = \pi.$$ 

It follows that $\pi^2 = \frac{1}{|G|} \sum_{h \pi} h \pi = \frac{1}{|G|} \sum \pi = \pi$. So $\pi$ is a linear projector. Let us determine its image. If $v \in C_V(G)$ then $\pi(v) = v$. Conversely, if $v \in V$, then for any $h \in G$ one has $h \pi(h) = \pi(v)$, so $\pi(v) \in C_V(G)$.

Thus $\pi$ is a projector with image $C_V(G)$ (we do not care for its kernel). Now for any projector, $\text{tr} \pi = \dim \text{im} \pi$; here this gives:

$$\dim C_V(G) = \dim \text{im} \pi = \text{tr} \pi = \frac{1}{|G|} \sum_{g \in G} \text{tr} g = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

(ii) Consider the $\mathbb{K}$-linear representation of $G$:

$$\text{Hom}_{\mathbb{K}}(V_1, V_2),$$

whose character is $\chi^* \chi_2$ by Proposition 4.3.3 (iv). We investigate the $G$-trivial space.

A $\mathbb{K}$-linear map $f : V_1 \to V_2$ is invariant under the action of $G$ if $f \text{iff } g \cdot f = f \text{iff } \rho_2(g) \circ f \circ \rho_1(g^{-1}) = f \text{iff } \rho_2(g) \circ f = f \circ \rho_1(g)$ iff $f$ is $G$-covariant. Hence:

$$C_{\text{Hom}_{\mathbb{K}}(V_1, V_2)}(G) = \text{Hom}_{\mathbb{K}(G)}(V_1, V_2).$$
5.3 Orthonormality

5.3.1. Lemma. Let \( G \) be a finite group and \( \mathbb{K} \) be a good field. Let \( V_1, V_2 \) be two \( \mathbb{K} \)-linear, finite-dimensional representations with characters \( \chi_1, \chi_2 \).

(i) If \( V_1 \) and \( V_2 \) are irreducible, then \( (\chi_1|\chi_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{otherwise.} \end{cases} \)

(ii) Every irreducible representation is determined by its character. If \( \text{char } \mathbb{K} = 0 \), this extends to every finite-dimensional representation.

(iii) The multiplicity of an irreducible representation \( T \) in \( \text{reg} \) is exactly \( \dim T \), viz.:

\[
\text{reg} \cong \bigoplus_{T \in \text{Irr}_G(\mathbb{K})} T^{\dim T}.
\]

Proof.

(i) Recall from Lemma 5.2.2 (ii) that \( \dim \operatorname{Hom}_{\mathbb{K}[G]}(V_1, V_2) = (\chi_1|\chi_2) \).

- If \( V_1 \) and \( V_2 \) are two non-isomorphic, irreducible representations, then \( \operatorname{Hom}_{\mathbb{K}[G]}(V_1, V_2) = \{0\} \) by Schur's Lemma, so \( (\chi_1|\chi_2) = 0 \).

- If \( V_1 \) and \( V_2 \) are irreducible and isomorphic, then on the one hand \( \chi_2 = \chi_1 \), and on the other hand, always by Schur's Lemma, \( \operatorname{Hom}_{\mathbb{K}[G]}(V_1, V_2) \cong \mathbb{K} \) is 1-dimensional. So \( (\chi_1|\chi_2) = \frac{|G|}{|\mathbb{K}|} = 1 \).

(ii) The irreducible case is trivial from (i). Now suppose \( \text{char } \mathbb{K} = 0 \). Let \( V \) be a finite-dimensional representation \( V \); then by Corollary 3.3.7, there are integers \( n_T \) such that:

\[
V \cong \bigoplus_{T \in \text{Irr}_G(\mathbb{K})} T^{n_T}.
\]

Then \( \chi_V = \sum n_T \chi_T \) where \( \chi_T \) is the character of \( T \). By linear independence of the orthonormal family, \( \chi_V \) completely determines the elements \( n_T \in \mathbb{K} \).

Now the ring morphism \( \mathbb{Z} \to \mathbb{K} \) is injective in characteristic 0, so \( \chi_V \) even determines the integers \( n_T \in \mathbb{Z} \). It therefore determines the isomorphism type.

(iii) A priori \( \text{reg} = \sum n_T \chi_T \) where the sum ranges over irreducible characters. We now determine the integers \( n_T \). By orthonormality, for any irreducible \( \psi \) one has:

\[
(\text{reg}|\psi) = \sum_{\chi \in \text{Irr}_G(\mathbb{K})} (n_T \chi|\psi) = \sum_{\chi \in \text{Irr}_G(\mathbb{K})} n_T \delta_{\chi, \psi} = n_{\psi} \delta_{\psi, \psi} = n_{\psi}.
\]

Now \( \text{reg} \) is the permutation character attached to the regular representation: if \( g \neq 1 \), then \( g \cdot e_h = e_{gh} \). So the matrix coding the action of \( g \) is a permutation matrix avoiding the diagonal, meaning \( \text{reg}(g) = 0 \). In particular, for any
character $\psi$ one has:

$$(\text{reg}|\psi) = \frac{1}{|G|} \sum_{g \in G} \text{reg}(g^{-1})\psi(g) = \frac{1}{|G|} \text{reg}(1)\psi(1) = \psi(1).$$

Applying to irreducible $\chi$, we get $n_\chi = \chi(1) = \dim \chi$, as wanted. \hfill \Box

Notice that the above also holds of $[\cdot \cdot]$ when working over (a subfield of) $\mathbb{C}$.

5.3.2. Remarks.

- Lemma 5.3.1 can be used to give a quick proof in characteristic 0 that a finite-dimensional representation $V$ is irreducible iff $V^*$ is. (This is true in any characteristic, but the general argument is more geometric: see exercise 2.5.5. Also see Remark 3.3.8.)

Indeed, $V$ is irreducible iff $(\chi_V|\chi_V) = 1$. This is because a priori, $V \cong \bigoplus T^{n_T}$; now $(\chi_V|\chi_V) = \sum n_T^2$ in $\mathbb{K}$. But in characteristic 0, this can equal 1 iff there is a unique non-zero $n_T$, which equals 1. This proves the claim.

Finally notice that $(\chi_V^*|\chi_V^*) = (\chi_V|\chi_V)$.

- The above proof does not work in positive characteristic $p > 0$ (even over a good field). Indeed, for $V = \text{triv} \oplus T^p$ one gets $(\chi_V|\chi_V) = 1$ but $V$ is certainly not irreducible. The reason is that here, $(\chi_V|\chi_V)$ is not an integer in $\mathbb{Z}$, but an element of the prime field of $\mathbb{K}$.

5.4 The space of class functions

We need one last fact to prove Theorem 5.1.1.

5.4.1. Proposition. Let $G$ be a finite group and $\mathbb{K}$ be a good field. Then $\text{Irr}_{\mathbb{K}}(G)$ spans $C_{\mathbb{K}}(G)$.

**Proof.** Let $\alpha: G \to \mathbb{K}$ be a class function. We shall prove that $\alpha = \sum_{\chi \in \text{Irr}_{\mathbb{K}}(G)} (\chi|\alpha) \chi$. Considering the difference $\alpha - \sum_{\chi \in \text{Irr}_{\mathbb{K}}(G)} (\chi|\alpha) \chi$, it suffices to prove that a class function orthogonal to all irreducible characters is trivial.

So let $\alpha$ be such. Let $V = \text{reg}$ be the regular representation and

$$f = \frac{1}{|G|} \sum_{g \in G} \alpha(g) g \in \text{End}_{\mathbb{K}}(V).$$

Since $\alpha$ is a class function, one can easily show that $f$ is $G$-covariant (see exercise 5.6.4), hence $f \in \text{End}_{\mathbb{K}(G)}(\text{reg})$.

Let $W \leq \text{reg}$ be any irreducible subrepresentation, with character $\chi$. By construction, $W$ is $f$-invariant; moreover $f$ remains an endomorphism of $W$. So by Schur’s Lemma there is $\lambda \in \mathbb{K}$ such that $f_{|W} = \lambda \text{Id}_W$. Then:

$$\lambda \dim W = \text{tr} f_{|W} = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \text{tr} g_{|W} = (\chi^*|\alpha).$$

Now $\chi^*$ is an irreducible character, so by assumption the above is 0; hence $\lambda = 0$ and
\[ f|_W = 0. \]
So \( f \) vanishes on all irreducible subrepresentations of \( G \). The latter is a direct sum of irreducible representations by Maschke's Theorem, so \( f = 0 \) globally. Finally
\[ f(e_i) = \frac{1}{|G|} \sum_{g \in G} \alpha(g) e_g = 0, \]
meaning that \( \alpha \) is identically 0. We are done. \( \square \)

5.5 Column orthogonality
Orthogonality has uncountable consequences. A useful tool is provided by the following lemma.

5.5.1 Lemma (column orthogonality). Let \( G \) be a finite group and \( \mathbb{K} \) be a good field. Then for any two conjugacy classes \( \gamma_1, \gamma_2 \), one has:

\[
\sum_{\chi \in \text{Irr}(G)} \chi(\gamma_1^{-1}) \chi(\gamma_2) = \begin{cases} 
\frac{|G|}{|\gamma_1|} & \text{if } \gamma_1 = \gamma_2, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. We use matrix theory. Let \( A = (\chi(y))_{y \in \gamma} \) be the character table. We also need the version with inverses: \( B = (\chi(y^{-1}))_{y \in \gamma} \). Last, let \( J \) be the diagonal matrix:

\[
J = \begin{pmatrix} 
\frac{|\gamma_1|}{|G|} & \cdots & \frac{|\gamma_1|}{|G|} \\
\vdots & \ddots & \vdots \\
\frac{|\gamma_2|}{|G|} & \cdots & \frac{|\gamma_2|}{|G|}
\end{pmatrix}.
\]

By orthogonality, the \((\chi_1, \chi_2)\)-entry of the product \( BJA^t \) is:

\[
(BJA^t)_{\gamma_1, \gamma_2} = \sum_{y \in \text{Conj}(G)} \chi_1(y^{-1}) \frac{|\gamma_2|}{|G|} \chi_2(y) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1}) \chi_2(g) = \delta_{\gamma_1, \gamma_2}.
\]

Hence \( BJA^t = I \) is the identity matrix. This implies \( AJB^t = I \), and \( B^t A = J^+ \). The latter gives, at \((\gamma_1, \gamma_2)\):

\[
\sum_{\chi \in \text{Irr}(G)} \chi(\gamma_1^{-1}) \chi(\gamma_2) = (B^t A)_{\gamma_1, \gamma_2} = (J^+)_{\gamma_1, \gamma_2} = \delta_{\gamma_1, \gamma_2} \frac{|G|}{|\gamma_2|}.
\]

\( \square \)

5.6 Exercises

5.6.1. Exercise. Let \( V \) be an irreducible representation and \( L \) be a 1-dimensional representation. Prove that \( V \otimes L \) is irreducible.

5.6.2. Exercise (column orthogonality). Let \( G \) be a finite group and \( \mathbb{K} \) be a good field.

1. Prove that \( x, y \) are conjugate iff \((\forall \chi \in \text{Irr}(G))(\chi(g) = \chi(h))\).
2. If \( \mathbb{K} \leq \mathbb{C} \), prove that the square matrix \( (\chi(y)) \cdot \sqrt{|\gamma|} \) is unitary.

5.6.3. Exercise. Let \( G \) be finite and \( V \) be a linear, complex, irreducible representation. Prove that \((\dim V)^2 \leq |G : Z(G)|\).
5.6.4. **Exercise.** Let $\alpha : G \to K$ be any map. For $V$ a $K$-linear representation of $G$, let:

$$f_{\alpha, V} = \frac{1}{|G|} \sum_{g \in G} \alpha(g) g : V \to V.$$ 

Prove that the following are equivalent:

(i) $\alpha$ is a class function;

(ii) for every representation $V$, $f_{\alpha, V}$ is a $G$-covariant endomorphism of $V$;

(iii) $f_{\alpha, \text{reg}}$ is a $G$-covariant endomorphism of $\text{reg}$.

5.6.5. **Exercise.** Let $G$ be a finite group and $K$ be a good field of positive characteristic. Let $V, V'$ be finite-dimensional representations with the same character. Prove that there are natural integers $n_T$ and $n'_T$ with $T \in \text{Irr}_K(G)$ such that:

- for each $T$, at least one of $n_T$ or $n'_T$ is 0,
- for each $T$, both $n_T$ and $n'_T$ are divisible by $\text{char } K$,
- one has:

$$V \oplus \bigoplus_T T^{n_T} = V' \oplus \bigoplus_T T'^{n'_T} \quad [K[G]\text{-Mod}].$$

(* ) 5.6.6. **Exercise.** Let $K$ be a good field. Let $V$ be a $K$-linear, finite-dimensional representation and $T \in \text{Irr}_K(G)$. Prove that the projector onto $\text{Iso}_T(V)$ parallel to the other terms $\bigoplus_{T' \neq T} \text{Iso}_{T'}(V)$ is given by:

$$\pi_T = \frac{\dim T}{|G|} \sum_{g \in G} \chi_T(g^{-1}) g.$$ 

6  **Computing character tables**

**Abstract.** A problem session.

We give a couple of character tables over $\mathbb{C}$. For each, I try to give a flow of natural arguments to determine it, and then a flow of natural comments on it. But there are many approaches to the same problem, so some of the remarks could be used in the determination process. The only way to read the notes for this section is by actually trying to construct the tables yourself.

**Sym(1)=Alt(2)**

- There is nothing to say before or after.

<table>
<thead>
<tr>
<th></th>
<th>$\text{Sym}(1)$</th>
<th>$\text{triv}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\ast]$</td>
<td>1</td>
<td>$\frac{1}{1}$</td>
</tr>
</tbody>
</table>
**Alt(3)**

- Alt(3) is abelian.
- By abelianity, conjugacy classes have one element, so there are three of them.
- By abelianity again, the irreducible representations have dimension 1, hence are simply morphisms Alt(3) → C\*\*. We need three of them (including triv).
- For a non-trivial morphism Alt(3) → C\*, the image of (123) in C\* must have order 3. Let \( j = e^{\frac{i\pi}{3}} \).

<table>
<thead>
<tr>
<th>Alt(3)</th>
<th>1 [\ast 1]</th>
<th>(123) [\ast 1]</th>
<th>(12) [\ast 1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>triv</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>( j )</td>
<td>( j^2 )</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>1</td>
<td>( j^2 )</td>
<td>( j )</td>
</tr>
</tbody>
</table>

- Here the orthogonality relations essentially reduce to: \( 1 + 1 + 1 = 3 \), and \( 1 + j + j^2 = 0 \).
- Observe how \( \chi_1^* = \psi_1 = \chi_1 \otimes \chi_1 \).

**Sym(3)**

- There are three conjugacy classes and we easily find their cardinalities.
- Therefore there are three irreducible representations.
- In addition to triv, there is the signature representation sign. We miss one more.
- Let \( d \) be its dimension. Since \((\dim \text{triv})^2 + (\dim \text{sign})^2 + d^2 = |\text{Sym}(3)| = 6\), the last irreducible representation is 2-dimensional. Call it \( \chi_3 \).
- One can predict that \( \chi_3 \) is real-valued, and vanishes on (12). Indeed:
  - By exercise 2.5.5 or Remark 5.3.2, \( \chi_3^* \) is an irreducible, 2-dimensional representation. But only \( \chi_1 \) is irreducible and 2-dimensional, so \( \chi_3^* = \chi_1 \), which means it is real-valued.
  - \( \chi_3 \otimes \text{sign} \) is an irreducible, 2-dimensional representation (exercise 5.6.1), so \( \chi_3 \otimes \text{sign} = \chi_3 \). But \( \text{sign}((12)) = 1 \), so \( \chi_3((12)) = 0 \).

One could use orthogonality to find \( \chi_3((123)) \), but geometry is more natural.
- Let \( \text{perm}_3 \) be the permutation representation, which is 3-dimensional. Clearly \( \text{perm}_3((1)) = 3 \), \( \text{perm}_3((12)) = 1 \), and \( \text{perm}_3((123)) = 0 \). Thus, \( \left( \text{perm}_3 | \text{perm}_3 \right) = \frac{1}{2}(9 \times 1 + 1 \times 3 + 0 \times 2) = 2 \). So \( \text{perm}_3 \) is the sum of two irreducible representations. One of them must have dimension 2: thus \( \chi_3 \) is a subrepresentation of \( \text{perm}_3 \).
- Of course \( e_1 + e_2 + e_3 \in \text{perm}_3 \) is fixed by Sym(3), so \( \text{perm}_3 \) contains a copy of triv.
- By the above, \( \chi_3 = \text{perm}_3 - \text{triv} \), and we get the table.

<table>
<thead>
<tr>
<th>Sym(3)</th>
<th>1 [\ast 1]</th>
<th>(12) [\ast 3]</th>
<th>(123) [\ast 2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>triv</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sign</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
Let us check orthogonality: $\bullet 1^2 \times 1 + 1^2 \times 3 + 1^2 \times 2 = 6$ expresses $(\text{triv}|\text{triv}) = 1$; $\bullet 1^2 \times 1 + (-1)^2 \times 3 + 1^2 \times 2 = 6$ expresses $(\text{sign}|\text{sign}) = 1$; $\bullet 1 \times 1 + (-1) \times 3 + 1 \times 2 = 0$ expresses $(\text{triv}|\text{sign}) = 0$; $\bullet$ and so on.

As predicted, $\chi_3 = \chi_3^* = \chi_3 \cdot \text{sign}$.

**Alt(4)**

One should be careful with conjugacy classes. Although $(123)$ and $(132)$ are conjugate in $\text{Sym}(4)$, the $\text{Sym}(4)$-conjugacy class 'breaks' into two when going down to $\text{Alt}(4)$. This produces two $\text{Alt}(4)$-classes of the same size. With this in mind, or just remembering $\text{Alt}(4) = C_2 \times C_3$, we find 4 conjugacy classes and count their elements.

There are 4 irreducible representations and triv is one of them. (For $\text{Alt}(n)$, by definition sign = triv.) We need three more.

Let $K = \{1, (12)(34), (13)(24), (14)(23)\} \leq \text{Alt}(4)$, the (very important) sub-group of bitranspositions. It is normal and $\text{Alt}(4)/K$ has order 3, hence is abelian. Now each representation of $\text{Alt}(4)/K$ gives one of $\text{Alt}(4)$ (by letting $K$ act trivially).

This way we gain two 1-dimensional representations of $\text{Sym}(4)$: just 'lifting' those of $\text{Sym}(4)/K \cong \text{Sym}(3)$. Call them $\chi_i$ and $\psi_i$.

We need one more. Its dimension satisfies $3 + d^2 = 12$, so the dimension is 3.

Of course it is $\text{perm}_4 - \text{triv}$, which is indeed irreducible by computation.

<table>
<thead>
<tr>
<th>$\text{Alt}(4)$</th>
<th>1 $\times 1$</th>
<th>$(123) \times 4$</th>
<th>$(132) \times 4$</th>
<th>$(12) \times (34) \times 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>triv</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>$j$</td>
<td>$j^2$</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_1 = \chi_1^*$</td>
<td>1</td>
<td>$j^2$</td>
<td>$j$</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Orthogonality can be checked, or simply deduced. Indeed, since $\chi_3 = \text{perm}_4 - \text{triv}$ is irreducible, it must be the missing irreducible character. By orthogonality, we know it is orthogonal to the others.

Here is the geometric realisation of $\chi_3$.

Consider the tetrahedron centered at the origin, whose vertices are:

$v_1 = (1,1,1)$, $v_2 = (1,-1,-1)$, $v_3 = (-1,1,-1)$, $v_4 = (-1,-1,1)$.

$\text{Alt}(4)$ acts on this tetrahedron. It is a good exercise to explicitly write the matrices in $\{e_1, e_2, e_3\}$, but we simply determine the character—in the basis $\{v_1, v_2, v_3\}$, where computations are easy:

- $(123)$ is a circular permutation of the same basis; the trace is 0. The same applies to its inverse.
- Now $(12)(34)$ swaps $v_1$ and $v_2$, but takes $v_3$ to $v_4 = -v_1 - v_2 - v_3$. So the trace is $-1$.

And we retrieve $\chi_3$. This geometric argument gives it directly, and not as a quotient of $\text{perm}_4$.
Sym(4)

- There are 5 conjugacy classes.
- We already know two irreducible representations: triv and sign.
- Now \( \text{perm}_4 – \text{triv} \) is again irreducible, and we can compute its character \( \chi_3 \).
- Then \( \chi_3^\ast = \chi_3 \), but \( \chi_3 \text{sign} \neq \chi_3 \), so we just produced a fourth irreducible representation.
- Computation reveals that the missing one has dimension 2. One can predict \( \chi_3^\ast = \chi_3 \) (real values) and \( \chi_3 \text{sign} = \chi_2 \) (so it vanishes where \( \text{sign} = -1 \)). One could then use orthogonality to compute the missing values.
- Now \( K = \{1, (12)(34), (13)(24), (14)(23)\} \) remains normal in Sym(4). This is most remarkable as normality is not transitive in general. Here, \( K = \text{Alt}(4)^1 \) and \( \text{Alt}(4) = \text{Sym}(4)^\prime \) are so-called characteristic subgroups, and being characteristic is transitive.
- Then \( \text{Sym}(4)/K \cong \text{Sym}(3) \). It suffices to extend the irreducible 2-dimensional representation of Sym(3) by letting \( K \) act trivially. Now in any isomorphism \( \text{Sym}(4)/K \cong \text{Sym}(3) \), the image of a 4-cycle becomes a transposition, while a bitransposition becomes the identity: this gives the values of \( \chi_3 \).

<table>
<thead>
<tr>
<th>Sym(4)</th>
<th>1 [x1]</th>
<th>(12) [x6]</th>
<th>(123) [x8]</th>
<th>(1234) [x6]</th>
<th>(12)(34) [x3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>triv</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sign</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( \psi_3 = \chi_3 \otimes \text{sign} )</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

- Here again, the geometric interpretation of \( \chi_3 \) is by acting on the tetrahedron. In the notation above, \((12)\) swaps \( v_1 \) and \( v_2 \), but fixes \( v_3 \) (and \( v_4 \)). So \( \chi_3((12)) = 1 \). Likewise, \((1234)\) takes \( v_1 \) to \( v_2 \), \( v_2 \) to \( v_3 \), and \( v_3 \) to \( v_4 = -v_1 - v_2 - v_3 \); thus \( \chi_3((1234)) = -1 \).
- The geometric interpretation of \( \chi_3 \) is different. Since \( K \leq \text{Sym}(4) \), there is a conjugation action on the set \( X = K \setminus \{1\} \). It is transitive. This gives us a 3-dimensional permutation representation.

Whenever one has a permutation representation, the vector \( \sum_X x \epsilon_x \) is fixed by \( G \), so here \( \text{perm}_X \) contains (at least) a copy of triv.

One checks that \( \chi_3 = \text{perm}_X - \text{triv} \).

Alt(5)

- Since Alt(5) is simple, the situation is completely different now (and interesting at last).
- In Alt(5), \((123)\) is conjugate to its inverse \((132)\). One may not use \((23) \notin \text{Alt}(5)\) to perform this conjugation, but \((23)(45) \in \text{Alt}(5)\) does as well.
• However, \((12345)\) is not \(\text{Alt}(5)\)-conjugate to its inverse \((15432)\); the \(\text{Sym}(5)\)-class of 5-cycles breaks into two \(\text{Alt}(5)\)-classes of equal size.

• In addition to \text{triv}, one easily finds the irreducible character \(\text{perm} - \text{triv}\); it has dimension 4. We need three more, and number theory gives dimensions 3, 3, 5.

• There are no subgroups of index 3 or 4, so actions on coset spaces are limited. Also, the action on cosets of \(\text{Alt}(4) \leq \text{Alt}(5)\) is equivalent to \text{perm}; this gives nothing new.

• An educated guess and 5-dimensionality suggest to look for a permutation representation on 6 elements, and Sylow theory provides one.

• \(\text{Alt}(5)\) has exactly 6 Sylow 5-subgroups, all of order 5.

• Let \(P = \{(12345)\}\); this is a Sylow 5-subgroup. Then \([G : N_G(P)] = 6\) so \(N_G(P)\) has order 10.

• This implies that in the conjugation action, an element of order 3 fixes no Sylow 5-subgroup.

• \([N_G(P)] = 10\) also implies that \(P\) is acted on by (the group generated by) a bitransposition; in abstract terms, \(N_G(P) \approx C_3 \times C_2\).

• The bi-transposition \(12)(34)\) fixes exactly two Sylow 5-subgroups: the one generated by \((1234)\), and the one generated by \((12453)\). (This can be seen because it inverts said generators.) But it fixes no other Sylow 5-subgroup and this remains to be seen.

• A good approach is by Burnside’s classical ‘fixed point formula’. For any finite group action \(G \sim X\), one has:

\[
\sum_G \# \text{Fix} g = \sum_{(G,X)} 1_{G \cdot x = x} = \sum_X |\text{Stab}_G(x)|.
\]

Here, the identity fixes 6 elements, a 3-cycle fixes 0, a 5-cycle fixes 1 (because if a 5 element normalises a Sylow 5-subgroup, it is already inside). There remains \(15 \cdot \# \text{Fix}(12)\) \((34) = 60 - 6 - 12 - 12 = 30\), and therefore \# \text{Fix}(12)\) \((34) = 2\).

• This gives us \(\text{perm}_{Syl5}\), and we can check that \(\chi_5 = \text{perm}_{Syl5} - \text{triv}\) is irreducible and 5-dimensional.

• We miss two more irreducible characters, say \(\varphi_3\) and \(\varphi_1'\). We do not know whether they will be complex-conjugate, or if each will be self-dual (real-valued). We then use brute force: orthogonality relations.

• Let \(a = \varphi_3((123))\), \(b = \varphi_3((12)(34))\), \(c = \varphi_3((12345))\) and \(d = \varphi_3((15432))\). Define \(a', b', c', d'\) similarly.

• By orthogonality:

\[
\begin{align*}
o &= |G|(\text{triv}|\varphi_3) = 3 + 20a + 15b + 12c + 12d \\
o &= |G|(\chi_4|\varphi_3) = 12 + 20a - 12c - 12d \\
o &= |G|(\chi_5|\varphi_3) = 15 - 20a + 15b.
\end{align*}
\]

This immediately yields \(a = 0\), then \(b = -1\), and \(c + d = 1\).
• We also have:
\[ 60 = |G| (\varphi_3 \varphi_3) = 9 + 15 + 12c^2 + 12d^2, \]
which gives \( c^2 + d^2 = 3 \).

• Together with \( c + d = 1 \), this solves into:
\[ \{ c, d \} = \left\{ \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right\} = \{ c', d' \}, \]
and we finally get the character table.

<table>
<thead>
<tr>
<th>( \text{Alt}(5) )</th>
<th>1 ([x_{1}])</th>
<th>(123) ([x_{20}])</th>
<th>(12)(34) ([x_{15}])</th>
<th>(12345) ([x_{12}])</th>
<th>(15432) ([x_{12}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{triv} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \varphi_3 )</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>( \frac{1 + \sqrt{5}}{2} )</td>
<td>( \frac{1 - \sqrt{5}}{2} )</td>
</tr>
<tr>
<td>( \varphi_4 )</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>( \frac{1 - \sqrt{5}}{2} )</td>
<td>( \frac{1 + \sqrt{5}}{2} )</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

• Although \( \varphi_3 \) and \( \varphi_4 \) are self-dual, they are indeed related by a Galois action (but not that of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \), which is generated by complex conjugation).

• Representations \( \varphi_3 \) and \( \varphi_4 \) arise in nature, as embeddings \( \text{Alt}(5) \hookrightarrow \text{SO}_5(\mathbb{R}) \). They are the symmetry groups of the regular icosahedron/dodecahedron.\(^6\)

6.1 Exercises

6.1.1. Exercise. Give the character tables of the group of isometries of the square (viz. \( G_{2,s} \)).
Same question with the group of basic quaternions (viz. \( \mathbb{H}_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \)).
Note. Thus, two non-isomorphic groups can have the same character table. \(^6\)

6.1.2. Exercise. Let \( G \) be the group of isometries of the Euclidean cube. Give its character table.

6.1.3. Exercise. Let \( G_{2,n} \) be the dihedral group of order \( 2n \), viz. the group of transformations of a regular \( n \)-gon in the usual plane. Give its character table.

6.1.4. Exercise. Compute the character table of \( \text{Sym}(5) \). You should find:

<table>
<thead>
<tr>
<th>( \text{Sym}(5) )</th>
<th>1 ([x_{1}])</th>
<th>(12) ([x_{10}])</th>
<th>(123) ([x_{10}])</th>
<th>(1234) ([x_{15}])</th>
<th>(12)(34) ([x_{24}])</th>
<th>(12)(345) ([x_{24}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{triv} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \text{sign} )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_4 \cdot \text{sign} )</td>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_5 \cdot \text{sign} )</td>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

\(^1\)I have to teach this in the Village someday.
\(^6\)Thanks to Baran Çetin for pointing this out.
7 Number-theoretic aspects

Abstract. From here on we shall be working in characteristic 0. The kernel of a representation is often called the character kernel ($\S$7.1); every normal subgroup is an intersection of irreducible character kernels. We then move to using algebraic number theory in character theory ($\S$7.2). A first application is a theorem by Burnside: the dimension of an irreducible complex representation divides $|G|$ ($\S$7.3).

7.1 Character kernels and their intersections

7.1.1 Definition. Let $G$ be a group and $\mathbb{K}$ be a field. Let $(V, \rho)$ be a finite-dimensional, $\mathbb{K}$-linear representation with character $\chi$. We let $\ker \chi = \ker \rho$ and call it the kernel of $\chi$.

This is a slight abuse of terminology since $\chi$ itself is not a morphism.

7.1.2 Lemma. Let $G$ be a finite group and $\mathbb{K}$ be a field. Let $(V, \rho)$ be a finite-dimensional, $\mathbb{K}$-linear, representation with character $\chi$.

(i) $\ker \chi$ is a normal subgroup of $G$.

(ii) Suppose that $\mathbb{K}$ has characteristic 0. Then $\ker \chi = \{ g \in G : \chi(g) = \chi(1) \}$.

Proof. (i) is completely obvious since $\ker \chi = \ker \rho$, a kernel in the usual sense. So we prove (ii). If $g \in \ker \chi$, then $\rho(g) = \text{Id}_V$ so $\chi(g) = \text{tr} \text{Id}_V = \chi(1)$. Conversely suppose $\chi(g) = \chi(1)$; we must prove $\rho(g) = \text{Id}_V$.

As above, $\chi(1) = \text{tr} \text{Id}_V = \dim V$. Also, $g$ has finite order, so there is an integer $k$ with $g^k = 1$, and $\rho(g)^k = \text{Id}_V$. Therefore all eigenvalues of $g$, even in an algebraic closure, must satisfy this equation.

We finish the proof with $\mathbb{K} = \mathbb{C}$ (see Remark 7.1.3). Then $\rho(g) \in \text{GL}(V)$ is an element of finite order say $k$. Since the polynomial $X^k - 1$ is split with simple roots, $\rho(g)$ is diagonalisable. Moreover, all its eigenvalues $\lambda_1, \ldots, \lambda_{\dim V}$ satisfy $\lambda_i^k = 1$, so they lie on the unit circle. By assumption, their sum is $\dim V$. This is possible only if each $\lambda_i = 1$. So $\rho(g)$ diagonalises to the identity, implying $\rho(g) = \text{Id}_V$, as wanted.

7.1.3 Remark. It is enough to have char $\mathbb{K} = 0$. Indeed, all coefficients and eigenvalues of $\rho(g)$ will live in an algebraic extension of $\mathbb{Q}$, so all in $\mathbb{Q} \subseteq \mathbb{C}$, and we safely conduct the argument there.

However (ii) no longer holds in non-zero characteristic. As opposed to many results, this one already fails in good fields of positive characteristic.

7.1.4 Theorem. Let $G$ be a finite group and $\mathbb{K}$ be an algebraically closed field of characteristic 0.

(i) $\cap_{\chi \in \text{Irr}(G)} \ker \chi = \{ 1 \}$.

(ii) Let $N \triangleleft G$ be a subgroup. Then $N \not\triangleleft G$ iff there is $J \subseteq \text{Irr}(G)$ such that $N = \cap_{\chi \in J} \ker \chi$.

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Proof.

(i) Let $K = \bigcap_{\chi \in \text{Irr}_G} \ker \chi$, an intersection of normal subgroups. We must show $K = \{1\}$. Let us fix some notation. First, let $\pi_K : G \to G/K$ be the canonical projection.

Let $\chi \in \text{Irr}_G(G)$. Then $\chi$ is the character of some irreducible, complex, linear representation $(V_{\chi}, \rho_{\chi})$, viz. we have a morphism $\rho_{\chi} : G \to \text{GL}(V_{\chi})$. Since $K \leq \ker \rho_{\chi}$, one may factor and consider:

$$\hat{\rho}_{\chi} : G/K \to \text{GL}(V_{\chi}).$$

By definition, $\rho_{\chi} = \hat{\rho}_{\chi} \circ \pi_K$.

We claim that $\hat{\rho}_{\chi}$ is an irreducible representation of $G/K$. This is obvious since $K$ acts trivially on $V_{\chi}$, so $G$-invariant subspaces are the same as $G/K$-invariant subspaces.

Let $\bar{\chi}$ be the character of $\hat{\rho}_{\chi}$. Thus $\bar{\chi} \in \text{Irr}_{G/K}(G/K)$. Again, $\chi = \bar{\chi} \circ \pi$. It follows that if $\chi_1 \neq \chi_2$ in $\text{Irr}_G(G)$, then $\bar{\chi}_1 \neq \bar{\chi}_2$ in $\text{Irr}_{G/K}(G/K)$. So characters $\bar{\chi}$ for $\chi \in \text{Irr}_G(G)$ are distinct elements of $\text{Irr}_{G/K}(G/K)$, meaning $\{\bar{\chi} : \chi \in \text{Irr}_G(G)\} \subseteq \text{Irr}_{G/K}(G/K)$.

By the orthogonality relations:

$$|G| = \sum_{\chi \in \text{Irr}_G(G)} \chi(1)^2 = \sum_{\chi \in \text{Irr}_G(G)} \bar{\chi}(1)^2 \leq \sum_{\psi \in \text{Irr}_{G/K}(G/K)} \psi(1)^2 = |G/K|,$$

which proves $|K| = 1$, as desired.

(ii) The converse implication is obvious, so suppose $N \trianglelefteq G$. Let $\pi_N : G \to G/N$ be the canonical projection. We consider $\text{Irr}_{G/N}(G/N)$, whose elements are the irreducible characters $\psi$, attached to morphisms $\sigma_N : G/N \to \text{GL}(V_{\psi})$.

For $\psi \in \text{Irr}_{G/N}(G/N)$, let:

$$\rho_{\psi} = \sigma_N \circ \pi_N : G \to \text{GL}(V_{\psi}).$$

We claim that $\rho_{\psi}$ is an irreducible representation of $G$. Indeed, a $G$-invariant subspace of $V_{\psi}$ is also $G/N$-invariant, hence $\{0\}$ or $V_{\psi}$ by irreducibility of $\sigma_N$.

Let $\hat{\psi}$ be the character of $\rho_{\psi}$. Thus $\hat{\psi} \in \text{Irr}_{G/N}(G/N)$. Again, $\psi = \hat{\psi} \circ \pi_N$. It follows that if $\psi_1 \neq \psi_2$ in $\text{Irr}_{G/N}(G/N)$, then $\hat{\psi}_1 \neq \hat{\psi}_2$ in $\text{Irr}_{G/N}(G/N)$. (This is because $\pi_N$ is onto.) So $J = \{\hat{\psi} : \psi \in \text{Irr}_{G/N}(G/N)\}$ is a family of irreducible characters of $G$.

We claim that $N = \bigcap_{\hat{\psi} \in J} \ker \chi$. Indeed,

$$\bigcap_{\chi \in J} \ker \chi = \bigcap_{\psi \in \text{Irr}_{G/N}(G/N)} \ker \rho_{\psi} = \bigcap_{\psi \in \text{Irr}_{G/N}(G/N)} \ker(\sigma_N \circ \pi_N) = \pi_N^{-1}\left(\bigcap_{\psi \in \text{Irr}_{G/N}(G/N)} \ker \sigma_{\psi}\right).$$

By (i) applied to $G/N$, the latter intersection is $\{1 \mod N\}$, so $\bigcap_{\chi \in J} \ker \chi = \pi_N^{-1}(1) = N$, as wanted. 

\hfill \Box
7.2 Algebraic integers

7.2.1. Definition. Let \( R \) be a ring with 1. An element \( x \in R \) is integral (over \( \mathbb{Z} \)) if there is a polynomial \( P \) \( \in \mathbb{Z}[X] \) with leading coefficient 1 such that \( P(x) = 0 \).

One often denotes by \( \mathcal{O}_R \) the set of integral elements of \( R \).

7.2.2. Remark. Since \( R[x] \) is always commutative, commutativity of \( R \) is not required for the general definition. (But one needs 1 \( \in R \) for the definition.)

If \( R = \mathbb{C} \), one calls \( x \) an algebraic integer; we simply write \( \mathcal{O} = \mathcal{O}_\mathbb{C} \).

7.2.3. Remark. Algebraic integers are not to be mistaken with algebraic numbers, where the condition on the leading coefficient is removed. Algebraic numbers exactly form the field \( \overline{\mathbb{Q}} \); but \( \mathcal{O} \) is a proper subring of \( \overline{\mathbb{Q}} \).

7.2.4. Example. \( \sqrt{2} \) is an algebraic integer; \( \frac{1}{2} \) is not.

7.2.5. Proposition. If \( R \) is a commutative ring with unit, then \( \mathcal{O}_R \) is a subring of \( R \).

Proof. Clearly 0, 1 are in \( \mathcal{O}_R \), and \( \mathcal{O}_R \) is closed under \( - \); so we need closedness under \( + \) and \( \cdot \).

7.2.6. Lemma. Let \( x \in R \). Then \( x \in \mathcal{O}_R \) iff \( \mathbb{Z}[x] \) is finitely generated as an abelian group.

Commutativity of \( R \) is not required in the Lemma, since \( R[x] \) always is commutative.

Proof. If \( x \) is integral and \( P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X] \) vanishes at \( x \), then clearly \( \mathbb{Z}[x] \) is generated by \( \{1, x, \ldots, x^{n-1}\} \) as an abelian group.

We prove the converse. For \( n \in \mathbb{N} \), let \( R_n \) be the abelian group generated by \( \{1, \ldots, x^{n-1}\} \). Then \( (R_n) \) is an ascending chain of abelian subgroups with union \( \mathbb{Z}[x] \). But the latter is finitely generated, so there is \( n \) such that \( R_n \) contains all generators. Then \( R_{n+1} = R_n \). This implies \( x^{n+1} \in R_n \), so \( x \) is integral.

If \( x \) and \( y \) are algebraic integers, then \( \mathbb{Z}[x] \) and \( \mathbb{Z}[y] \) are finitely generated, and so is their tensor product \( \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[y] \), which maps onto \( \mathbb{Z}[x, y] \). The latter contains \( \mathbb{Z}[x + y] \) and \( \mathbb{Z}[x \cdot y] \). It is not true in general that subgroups of finitely generated groups are finitely generated, but this holds of abelian groups.

7.2.7. Remark. Proposition 7.2.5 requires commutativity of \( R \). The reason is that the proof uses that \( \mathbb{Z}[x, y] \) is an image of \( \mathbb{Z}[x] \times \mathbb{Z}[y] \), which holds only if \( x \) and \( y \) commute. See Remark 7.3.2.

7.2.8. Lemma. \( \mathcal{O} \cap \mathbb{Q} = \mathbb{Z} \).

Proof. The converse inclusion is obvious. Suppose \( x = \frac{p}{q} \in \mathcal{O} \) where \( p \) and \( q \) are coprime. Since \( x \in \mathcal{O} \), there is a polynomial \( P = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X] \) with leading coefficient 1 such that \( P(x) = 0 \). Multiplying by \( q^n \) one has:

\[
p^n + a_{n-1}qp^{n-1} + \cdots + a_0q^n = 0
\]

Since \( p \) and \( q \) are coprime, one has \( q = 1 \), so \( x \in \mathbb{Z} \).
Characters and algebraic integers.

7.2.9. Proposition. Let $G$ be a finite group and $\chi$ be a complex character. Then $\chi$ takes values in $\mathbb{O}$.

Proof. Write $\rho(g)$ in diagonal form. Diagonal entries are roots of unity, hence algebraic integers. So their sum $\chi(g)$ is an algebraic integer. $\square$

7.3 A Theorem of Burnside

7.3.1. Theorem. Let $V$ be a complex, irreducible representation of a finite group $G$. Then $\dim V$ divides $|G|$.

Proof. Let $V$ be an irreducible representation of $G$ and let $\chi$ be its character; we wish to prove that $\dim V$ divides $|G|$.

Let $\gamma$ be a conjugacy class and $f_\gamma = \sum_{g \in \gamma} \rho(g) : V \to V$. Then $f_\gamma$ is $G$-covariant, so by Schur’s Lemma there is $\lambda_\gamma \in \mathbb{C}$ with $f_\gamma = \lambda_\gamma \text{Id}$. Furthermore, taking the trace one has:

$$\lambda_\gamma \dim V = \sum_{g \in \gamma} \chi(g) = \#\gamma \cdot \chi(\gamma).$$

Step 1. $\lambda_\gamma \in \mathbb{O}$.

Verification. While it is now clear that $\chi(\gamma) \in \mathbb{O}$ and $\#\gamma \cdot \chi(\gamma) \in \mathbb{O}$, we even want $\lambda_\gamma = \frac{\#\gamma \cdot \chi(\gamma)}{\dim V} \in \mathbb{O}$. Division is not permitted so there is something to prove.

One could argue as follows in the group ring $\mathbb{Z}[G]$ (§13):

Let $e_\gamma = \sum_{g \in \gamma} g \in \mathbb{Z}[G]$. Then $e_\gamma \in \mathbb{Z}(\mathbb{Z}[G])$. The latter is a commutative ring, and finitely generated as a group. So $e_\gamma \in \mathbb{O}(\mathbb{Z}[G]) \leq \mathbb{O}_{\mathbb{Z}[G]}$. Thus $\rho(e_\gamma) = f_\gamma = \lambda_\gamma \text{Id}_V$ is an algebraic integer of $\text{End}(V)$, and $\lambda_\gamma \in \mathbb{O}$.

But we have not introduced the algebraic object $\mathbb{Z}[G]$. (Also, be careful; see Remark 7.3.2.) We therefore give an ad hoc argument which will reappear in Lemma 8.2.1.

Let $\delta$ be another conjugacy class. For $x \in G$ let $X_{\gamma, \delta} = \{(g, h) \in \gamma \times \delta : x = gh\}$. Then:

$$f_\gamma f_\delta = \sum_{g \in \gamma} \sum_{h \in \delta} \rho(gh) = \sum_{x \in G} \left( \sum_{(g, h) \in X_{\gamma, \delta}} \rho(x) \right) = \sum_{x \in G} \#X_{\gamma, \delta} \rho(x).$$

Actually $\#X_{\gamma, \delta}$ depends only on the conjugacy class of $x$. Indeed, if $x' = x^y \in x^G$, then the map $(g, h) \mapsto (g^y, h^y)$ defines a bijection $X_{\gamma, \delta} \simeq X'_{\gamma, \delta}$. So the integer $\#X_{\gamma, \delta}$ is constant on $x^G$. Therefore there are integers $n_{\gamma, \delta, x}$ such that:

$$f_\gamma f_\delta = \sum_{ee \in \text{Conj}(G)} \sum_{x \in \xi} n_{\gamma, \delta, x} \rho(x) = \sum_{ee \in \text{Conj}(G)} n_{\gamma, \delta, e} f_\epsilon.$$

Returning to the irreducible representation, this implies:

$$\lambda_\gamma \lambda_\delta = \sum_{ee \in \text{Conj}(G)} n_{\gamma, \delta, e} \lambda_\epsilon.$$
Let $\Lambda$ be the column vector of the $\lambda_{\delta}$ for $\delta \in \text{Conj}(G)$. Let $A$ be the matrix with entries $(n_{\gamma, \delta, \varepsilon})$ for $\delta, \varepsilon \in \text{Conj}(G)$. Then varying $\delta$, the equations above rewrite:

$$A \cdot \Lambda = \lambda_{\gamma} \Lambda.$$ 

So $\lambda_{\gamma}$ is an eigenvalue of the integral matrix $A$, and therefore a solution of its characteristic polynomial. But the latter is in $\mathbb{Z}[X]$ and has leading coefficient $1$. Hence $\lambda_{\gamma}$ is an algebraic integer. 

\[\diamond\]

**Step 2.** A formula for $\frac{|G|}{\text{dim } V}$. 

**Verification.** Since $V$ is irreducible, $(\chi|\chi) = 1$. Therefore:

$$|G| = \sum_{g \in G} \chi(g^{-1}) \chi(g)$$

$$= \sum_{\gamma \in \text{Conj}(G)} \chi(g^{-1}) \chi(\gamma)$$

$$= \sum_{\gamma \in \text{Conj}(G)} \lambda_{\gamma} \dim V,$$

and

$$\frac{|G|}{\text{dim } V} = \sum_{\gamma \in \text{Conj}(G)} \chi(g^{-1}) \lambda_{\gamma}.$$ 

For each $\gamma$, we know that $\chi(g^{-1})$ is an algebraic integer (Proposition 7.2.9). Likewise, $\lambda_{\gamma}$ is an algebraic integer by Step 1. Since $\mathbb{O}$ is a ring, we find $|G|/\text{dim } V \in \mathbb{O}$. But obviously $|G|/\text{dim } V \in \mathbb{Q}$. By Lemma 7.2.8, $|G|/\text{dim } V$ is an integer. 

\[\diamond\]

7.3.2. Remark (if you already know the group algebra). It is the case that every $g \in G$ is an algebraic integer of $\mathbb{Z}[G]$. But it is not the case that every sum of $g$'s chosen at random is one. This fails because $\mathbb{Z}[G]$ is not a commutative ring. See exercise 7.4.5.

### 7.4 Exercises

#### 7.4.1. Exercise.** Let $G$ be a finite group. Prove the following equivalence:

(i) $G$ is simple;

(ii) for $\chi \in \text{Irr}_C(G)$, one has $\ker \chi = G$ or $\ker \chi = \{1\}$;

(iii) for $\chi \in \text{Irr}_C(G) \setminus \{\text{triv}\}$ and $g \in G \setminus \{1\}$, one has $\chi(g) \neq \chi(1)$.

#### 7.4.2. Exercise.** Let $G$ be a finite group.

1. Prove that $G$ is simple iff:

$$\forall \chi \in \text{Irr}_C(G) \forall g \in G \left[ (\chi(g) = \chi(1)) \rightarrow (\chi = \text{triv} \lor g = 1) \right].$$

\((*)\)

2. Devise a solubility test from the character table.

#### 7.4.3. Exercise.** Let $G$ be a finite group and $G \to \text{GL}(V)$ be a finite-dimensional, linear, complex representation with character $\chi$. 

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1. Prove that \(|\ker \chi| = \{ g \in G : |\chi(g)| = \dim V \} \) is a normal subgroup.

\(^*\)

2. Prove that \(\mathcal{G} : \ker \chi = \mathbb{Z}(G)\).

### 7.4.4. Exercise
Let \(G\) be a finite simple group. Prove that no irreducible complex representation has dimension 2. Hint: use Theorem 7.3.1 to prove that \(G\) has an involution.

### 7.4.5. Exercise
One needs to know or admit existence of the group ring \(\mathbb{Z}[G]\) (see § 13). Let \(G = \text{Sym}(3)\) and \(x = (12) + (23) \in \mathbb{Z}[G]\). Prove that \(x\) is not an algebraic integer of \(\mathbb{Z}[G]\).

### 8 Burnside’s \(p^a q^b\) theorem

**Abstract.** An application of character theory: Burnside’s \(p^a q^b\) theorem (§ 8.1). The proof uses a little algebraic number theory (§ 8.2) and is given in § 8.3.

#### 8.1 Statement

**8.1.1. Theorem.** Let \(G\) be a finite group of order \(p^a q^b\) where \(p, q\) are prime numbers. Then \(G\) is soluble.

One must recall the definition of a **soluble** group. Actually the proof also relies on **nilpotent** groups, and finite Sylow theory. The following facts will be required:

- every finite group has a \(p\)-subgroup of maximal order;
- every finite \(p\)-group is nilpotent.

Hence groups of order \(p^a\) are nilpotent, and groups of order \(p^a q^b\) are soluble. There is nothing to say about groups of order \(p^a q^b r\); for instance \(\text{Alt}(5)\) has order \(60 = 2^2 \cdot 3 \cdot 5\) but is simple.

**8.1.2. Remark.** We shall give a character-theoretic proof of the theorem. However, there exist character-free proofs; a full one which is not completely elementary, and two partial elementary proofs in the odd and even cases.\(^7\)

#### 8.2 A number-theoretic lemma

**8.2.1. Lemma.** Let \(G\) be a finite group; work over \(\mathbb{C}\). Let \((V, \rho, \chi) \in \text{Irr}_C(G)\) and \(g \in G\).

(i) The complex number:

\[
\left[ G : C_G(g) \right] \frac{\chi(g)}{\chi(1)}
\]

is an algebraic integer.

---

\(^7\)They are respectively:

(ii) If $[G : C_G(g)]$ and $\chi(1)$ are coprime, then $\chi(g) = 0$ or $\rho(g) \in \mathbb{C} \text{Id}_V$.

**Proof.**

(i) Throughout we work in $\text{End}_\mathbb{C}(V)$. For $y$ a $G$-conjugacy class let:

$$f_y = \sum_{g \in y} g.$$ 

Clearly $f_y \in \text{End}_{\mathbb{C}[G]}(V)$. By Schur’s Lemma, there is $\lambda_y \in \mathbb{K}$ such that $f_y = \lambda_y \text{Id}_V$. Therefore:

$$\lambda_y \chi(1) = \text{tr} f_y = \sum_{g \in y} \chi(g) = \# y \cdot \chi(y).$$

For $g \in y$, one has $\# y = [G : C_G(g)]$. Hence:

$$[G : C_G(g)] \frac{\chi(g)}{\chi(1)} = \lambda_y,$$

and it remains to prove that $\lambda_y$ is an algebraic integer.

We argue exactly like in the proof of Theorem 7.3.1, Step 1: there are integers $n_{y, \delta, \epsilon}$ such that:

$$f_y f_\delta = \sum_{c \in \text{Conj}(G)} n_{y, \delta, \epsilon} f_c.$$ 

This means $\lambda_y \lambda_\delta = \sum_{c \in \text{Conj}(G)} n_{y, \delta, \epsilon} \lambda_c$. Now let $A = (n_{y, \delta, \epsilon})_{\delta, \epsilon}$, a square matrix with integer entries, and let $A = (\lambda_\delta)_{\delta, \epsilon}$, a column vector with complex entries. Clearly $A \neq 0$. The above rewrites:

$$\lambda_y A = AA.$$ 

Hence $\lambda_y$ is an eigenvalue of $A$, and $\lambda_y$ is an algebraic number by the Cayley-Hamilton theorem.

(ii) By Bézout’s theorem, there are $a, b \in \mathbb{Z}$ such that:

$$a[G : C_G(g)] + b \chi(1) = 1,$$

which immediately yields $a[G : C_G(g)] \frac{\chi(g)}{\chi(1)} + b \chi(g) = \frac{\chi(g)}{\chi(1)}$. The terms of the left-hand member are algebraic integers, and therefore so is $\frac{\chi(g)}{\chi(1)}$. Now $\chi(g)$ is a sum of $\chi(1)$-many roots of unity. We finish with an algebraic lemma.

**8.2.2. Lemma.** Let $x_1, \ldots, x_n$ be complex roots of unity and $m = \frac{x_1 + \cdots + x_n}{n}$. If $m \in \mathbb{O}$, then $m = 0$ or $x_1 = \cdots = x_n = m$.

**Sketch of proof.** The proof uses a little Galois theory. Say all $x_i$ are $k$th roots of unity; let $\zeta = e^{\frac{2\pi i}{k}}$. Now let $\mathbb{F} = \mathbb{Q}(\zeta)$ and $\Sigma = \text{Gal}(\mathbb{F} : \mathbb{Q})$. By the fundamental theorem of Galois theory, $C_\mathbb{F}(\Sigma) = \mathbb{Q}$.

Let $q = \prod_{\sigma \in \Sigma} \sigma(m)$. Clearly $\Sigma$ maps $\mathbb{O}$ to $\mathbb{O}$; so $q \in \mathbb{O}$. But $q \in C_\mathbb{F}(\Sigma)$, so $q \in \mathbb{O} \cap \mathbb{Q} = \mathbb{Z}$. Now $\Sigma$ maps roots of unity to roots of unity. In particular for
every $\sigma \in \Sigma$, one has $|\sigma(m)| \leq 1$. Thus $|q| \leq 1$, and two cases remain.

- If $|q| = 0$ then $q = 0$, so one $\sigma(m)$ is zero. So is $m$.
- If $|q| = 1$, then $|m| = 1$ as well. A clear convexity argument gives that all $x_i$ are equal (and equal to $m$).

Let $\lambda_1, \ldots, \lambda_{\chi(1)}$ be the eigenvalues of $\rho(g)$. By Lemma, either their sum is 0, meaning $\chi(g) = 0$, or they are all equal, in which case $\rho(g) = \lambda \text{Id}_V$.

8.3 The main lemma, and proof of Burnside’s theorem

8.3.1. Lemma. Let $G$ be a finite group. Suppose there is a conjugacy class $\gamma \in \text{Conj}(G)$ with $\# \gamma$ a prime power. Then $G$ is not simple.

**Proof.** Let $g \in G$ be such that $\gamma = g^G$ has cardinality $p^k$ for some prime $p$ and $k > 0$. Suppose $G$ is simple. Then every non-trivial representation is injective.

Since $g \not\in 1^G = \{1\}$, by column orthogonality (Lemma 5.5.1) we have:

$$\sum_{\chi \in \text{Irr}(G)} \chi(1)\chi(g) = \sum_{\chi \in \text{Irr}(G)} \chi(1)\chi(g) = 0.$$  

Separating $\text{triv}$ from the sum and dividing,

$$\sum_{\chi \in \text{Irr}(G) \setminus \{\text{triv}\}} \frac{\chi(1)\chi(g)}{p} = -\frac{1}{p}.$$  

Now $\frac{1}{p}$ is a proper rational, so it is not an algebraic integer. Since a sum of algebraic integers is again an algebraic integer, there is $\chi \in \text{Irr}(G) \setminus \{\text{triv}\}$ such that $\frac{\chi(1)\chi(g)}{p}$ is not an algebraic integer; this certainly implies $\chi(g) \neq 0$.

Now $\chi(g)$ is an algebraic integer, so $p$ does not divide $\chi(1)$. In particular $\chi(1)$ and $\# g^G = |G : C_G(g)| = p^k$ are coprime. Moreover, $\chi(g) \neq 0$. By Lemma 8.2.1 (ii), there is $\lambda$ with $\rho(g) = \lambda \text{Id}_V$. This implies $\rho(g) \in Z(\rho(G))$. But $\rho$ is injective, so $g \in Z(G)$: a contradiction.

**Proof of Burnside’s $p^a q^b$ theorem.** Let $G$ be a counterexample of minimal order. If $a = 0$ or $b = 0$, then $G$ has order a prime power, so it is nilpotent: hence not a counterexample, a contradiction. Hence both $a$ and $b$ are non-zero.

If $G$ is not simple, then there is $\{1\} < N < G$. Notice that $N$ and $G/N$ still have order of the form $p^a q^b$. By minimality, both $N$ and $G/N$ are soluble; hence so is $G$, a contradiction. So $G$ is simple. If $Z(G) \neq \{1\}$ then by simplicity $Z(G) = G$ and $G$ is abelian: a contradiction.

Let $P < G$ be a Sylow $p$-subgroup; since $a > 0$, one has $P \neq \{1\}$. Since $P$ is a non-trivial, finite $p$-group, it has a non-trivial centre; let $g \in Z(P) \setminus \{1\}$. Then $P \leq C_G(g)$, so $[g^G]$ divides $q^b$. On the other hand $g \not\in Z(G) = \{1\}$. So $g^G \neq \{1\}$ is a conjugacy class of cardinality a prime power. By Lemma 8.3.1, $G$ is not simple, a contradiction.
Abstract. Induced representations (§ 9.1) construct representations of subgroups. The Frobenius formula (§ 9.2) is an explicit formula for induced characters. Its consequence, Frobenius reciprocity (§ 9.3), plays an important role in applied character theory.

Suppose $H \leq G$ are groups. In this section and the next, $c, d$ will stand for (left-)cosets of $G$ modulo $H$.

9.0.1. Notation. If $(V, \rho)$ is a representation of $G$, then the restriction $\rho|_H : H \to \text{GL}(V)$ define a representation of $H$, denoted by $\text{Res}^G_H(\rho)$.

(In general, irreducibility is not preserved.)

The whole section discusses one basic, converse, question. Suppose $(W, \sigma)$ is a representation of $H$. Does it come from some representation of $G$? We start with a basic lemma on 'coset geometry'.

9.0.2. Lemma. Let $H \leq G$ be groups and $c = aH$ be a (left-)coset of $H$. Let $g^c = \{ gb : b \in c \}$. Then $(gc = c)$ iff $(g^b \in H)$ iff $(g^c \subseteq H)$.

Proof.

• If $gc = c$, then $gaH = aH$ and $ga \in aH$, so there is $h \in H$ with $ga = ah$, viz. $g^a = a^{-1}ga = h \in H$.

• If $g^a \in H$ and $b \in c$, then there is $h \in H$ with $b = ah$. Hence $g^b = g^{ah} \in H^b = H$.

• If $g^c \subseteq H$, then for $b \in c$ there is $h \in H$ with $b^{-1}gb = g^b = h$, so $gb = bh$ and $gc = gbH = bhH = bH = c$.

9.1 Induced representations

Let $H \leq G$ be groups and $\sigma : H \to \text{GL}(W)$ be a representation of $H$. In general there is no $\rho : G \to \text{GL}(W)$ extending $\sigma$ (exercise 9.4.4). But if we allow for a larger vector space, an extension can be found. The present subsection describes this construction.

9.1.1. Definition. Let $H \leq G$ be groups and $\mathbb{K}$ be a field. Let $(W, \sigma)$ be a $\mathbb{K}$-linear representation of $H$. Construct a $\mathbb{K}$-linear representation of $G$ as follows.

• Let $\{ a_c : c \in G/H \}$ be a transversal of $H$ in $G$, viz. a set of representatives of the left-cosets, so that $G = \bigsqcup_{c \in G/H} a_c H$. We request $a_H = 1$.

• Let $V = \bigoplus_{c \in G/H} a_c W$ be a vector space obtained as a direct sum of $[G : H]$ copies of $W$.

• For $g \in G$ and $v = a_c w \in V$, first write $ga_c = a_d h$, then let:

$$g \cdot v = a_d (h \cdot w).$$

Extend linearly this action.

The resulting object is called the induced representation of $G$, denoted by $\text{Ind}^G_H W$. 47
(It is perfectly fine to leave $\mathbb{K}$ implicit in notation.)

9.1.2. Proposition.

(i) This is well-defined and does define a representation of $G$ in $V$.

(ii) $a_H W \leq V$ is $H$-invariant and $W = a_H W \ [\mathbb{K}[H]\text{-Mod}]$.

(iii) The $[\mathbb{K}[G]\text{-isomorphism type of the}]$ construction does not depend on the transversal chosen, provided $a_H = 1$.

(iv) If $V'$ is another representation of $G$ and $f: W \rightarrow \text{Res}_H^G V'$ is $H$-covariant, then there is a unique $\tilde{f}: \text{Ind}_H^G \rightarrow V'$ which is $G$-covariant and extends $f$.

Proof.

(i) By definition of a transversal, if $g \in G$ and $c \in G/H$, there is a unique pair $(a_d, h)$ such that $gac = a_dh$. So the construction is well-defined. Clearly each $g$ acts linearly. We now check that we have defined a morphism $G \rightarrow \text{GL}(V)$. Clearly 1 acts as the identity (this does not require $a_H = 1$ yet). Now let $g, g' \in G$ and $v \in V$. We must check $g(g'v) = (gg'v)$. By linearity, we may suppose $v = a_cw$ for some $a_c$ and $w \in W$.

By construction, $g' a_c = a_dh'$ and $ga_d = a_eh$ for cosets $d, e$. Altogether, this gives:

$$g'v = a_d(h'w),$$

and then:

$$g(g'v) = a_c(hh'w).$$

On the other hand, $(gg')a_c = g(g'a_c) = g(a_dh') = (ga_d)h' = a_c hh'$, so we also have:

$$(gg')v = a_c(hh'w).$$

This proves multiplicativity of the action. We have constructed a representation.

(ii) Let $h \in H$ and $v = a_H W$. Then $ha_H \in H$, so $a_{ha_H} = a_H h'$ for some $h'$. Then $h \cdot v = a_H(h'w) \in a_H W$, which is therefore $H$-invariant (this does not require $a_H = 1$ yet).

We now construct an isomorphism of $H$-representations $W \simeq a_H W$, using $a_H = 1$. Map $w$ to $\varphi(w) = a_H w$. Then for $h \in H$ one has $ha_H = a_H h$, so:

$$h \cdot \varphi(w) = h \cdot (a_H w) = a_H (hw) = \varphi(hw),$$

as wanted. (Actually this only requires $a_H \in Z(H)$.)

(iii) Suppose $\{b_c : c \in G/H\}$ is another transversal, also with $b_H = 1$. So our construction now comes in two flavours: $V_a$ and $V_b$. We must give an isomorphism of representations of $G$. For $c \in G/H$, one has $a_c H = b_c H$, so $\eta_c = b_c \cdot b_c^{-1} a_c \in H$.

To $v = a_c w$ associate $\varphi(v) = b_c(\eta_c w)$, and extend linearly. It clearly defines a linear isomorphism $V_a = V_b$. We contend it is a $G$-isomorphism. So let $g \in G$; by linearity, it is enough to prove $g \cdot \varphi(v) = \varphi(g \cdot v)$ for $v$ of the form $a_c w$. 

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Write \( ga_c = a_d h_1 \) and \( gb_c = b_d h_2 \) in obvious notation. Then:
\[
g \cdot \varphi(v) = g \cdot (b_c(\eta_c w)) = b_d(h_2 \eta_c w),
\]
while:
\[
\varphi(g \cdot v) = \varphi(a_d(h_1 w)) = b_d(\eta_d h_1 w).
\]
We must check equality. Indeed,
\[
b_d h_2 \eta_c = gb_c \eta_c = ga_c = a_d h_1 = b_d \eta_d h_1,
\]
so by definition of a transversal, \( d_i = d_i, \) and \( h_2 \eta_c = \eta_d h_1. \) So we are done.

(iv) Suppose \( \hat{f} : \text{Ind}^G_H \to V' \) is \( G \)-covariant. Then for \( c \in G/H \) and \( w \in W, \) one has:
\[
\hat{f}(a_c \cdot w) = \hat{f}(a_c(ah w)) = \hat{f}(a_c w) = a_c \hat{f}(w) = a_c f(w),
\]
so \( \hat{f}(a_c w) = a_c f(w) \). This guarantees uniqueness.

Conversely we let \( \hat{f}(a_c w) = a_c f(w) \) and extend linearly. This does define a linear map \( \hat{f} : \text{Ind}^G_H W \to V' \). We prove \( G \)-covariance on basic terms \( a_c w. \) Indeed, with \( ga_c = a_d h \) in obvious notation:
\[
\hat{f}(g \cdot (a_c w)) = \hat{f}(a_d hw) = a_d \hat{f}(hw) = a_d f(hw)
= a_d f(w) = ga_c f(w) = g \hat{f}(a_c w),
\]
proving \( G \)-covariance.

\subsection{Remarks.}

- As a consequence of Proposition 9.1.2 (iii), we may write \( \text{Ind}^G_H W = \bigoplus_{c \in G/H} c W \) with no mention of the transversal.
- (iv) is a universal property, conveniently described in terms of \textit{adjoint functors}.
- There is a tensor description of \( \text{Ind}^G_H W, \) but this course avoids tensoring over general rings.
- Recall that \( \text{Res}^G_H \) does not change the underlying vector space, but \( \text{Ind}^G_H \) does ('by a factor \([G : H]\']).

\section{Induced class functions and Frobenius formula}

If \( \beta \in \mathcal{C}_G(G) \) is a class function on \( G \) and \( H \leq G \) is a subgroup, we denote by \( \text{Res}^G_H \beta \) the restriction \( \beta|_H. \) Clearly \( \beta|_H \in \mathcal{C}_H(H) \); the operator \( \text{Res}^G_H : \mathcal{C}_G(G) \to \mathcal{C}_H(H) \) is clearly linear. We now define a 'converse' linear operator \( \text{Ind}^G_H : \mathcal{C}_H(H) \to \mathcal{C}_G(G) \). It is converse in a loose sense since in general, \( \text{Res} \text{Ind}_h = \alpha. \) (‘Adjoint’ would be more adapted.)

\subsection{Lemma.}

Let \( H \leq G \) be a pair of finite groups and \( \mathbb{K} \) be a field.

(i) For \( \alpha \in \mathcal{C}_H(H) \) a class function on \( H, \) the following definition makes sense:
\[
\text{Ind}^G_H \alpha = \sum_{c \in G/H} \alpha(g^{-1} c).
\]
(ii) If $W$ is a finite-dimensional, $\mathbb{K}$-linear representation of $H$ with character $\chi_W$, then:

$$\text{Ind}_{H}^{G} \chi_W = \chi_{\text{Ind}_{H}^{G} W}.$$  

'The character of the induced (representation) is the induced (function) of the character.'

Proof.

(i) Suppose $a, b \in c$; say $a_c = b_c \eta_c$ with $\eta_c \in H$. Then by Lemma 9.0.2:

$$g_c = c \iff g^{a_c} \in H \iff g^{b_c} \in H.$$  

So if $g_c = c$, then $\alpha(g^{a_c})$ and $\alpha(g^{b_c})$ both make sense. Moreover, $g^{b_c} \eta_c = g^{a_c}$, so $g^{a_c}$ and $g^{b_c}$ are $H$-conjugate.

Therefore always assuming $g_c = c$, and since $\alpha$ is a class function on $H$, we have:

$$\alpha(g^{b_c}) = \alpha(g^{a_c}).$$  

Hence $\alpha(g^{c})$ is well-defined regardless of the choice of the transversal: the formula makes sense.

(ii) Let $V = \text{Ind}_{H}^{G} W$, with character $\chi_V$. We prove $\chi_V = \text{Ind}_{H}^{G} \chi_W$.

Let $g \in G$. Recall that $V = \bigoplus_{c \in \mathbb{G}/H} a_c W$. Moreover, $g$ maps the space $a_c W$ to the space $a_d W$ for $ga_c = a_d h$. But when computing $\text{tr} g$, only $g$-invariant subspaces from the direct sum contribute, and:

$$\text{tr} g = \sum_{g(a_c W) \leq a_c W} \text{tr}(g_{\langle a, \rangle W}).$$

Notice that $ga_c W \leq a_c W$ iff there is $h \in H$ with $ga_c = a_h$ iff $g^{a_c} \in H$ iff $g_c = c$.

So:

- cosets $d$ with $gd \neq d$ do not contribute;
- cosets $c$ with $g_c = c$ contribute $\text{tr}(g_{\langle a, \rangle W})$.

In the latter case, $g$ acts on $a_c W$ like $h = g^{a_c}$. Therefore $\text{tr}(g_{\langle a, \rangle W}) = \chi_W(g^{a_c}) = \chi_W(g^c)$.

Hence:

$$\chi_V(g) = \text{tr}(g) = \sum_{g_c = c} \chi_W(g^c) = \text{Ind}_{H}^{G} \chi_W.$$

9.3 Frobenius reciprocity

Since we deal with two groups, it is convenient to denote $(\cdot | \cdot)_K$ the usual bilinear form with respect to $K \in \{H, G\}$.

9.3.1. Theorem (Frobenius reciprocity). Let $H \leq G$ be finite groups and $\mathbb{K}$ be a $G$-good field (hence $H$-good as well).
(i) Let $W$ be a $\mathbb{K}$-linear, finite-dimensional representation of $H$ and $V$ be one of $G$. Then:

$$
\left( \chi_{\text{Ind}_W^G} | \chi_{\text{Res}^V_V} \right)_G = \left( \chi_W | \chi_{\text{Res}^V_V} \right)_H.
$$

(ii) For $\alpha \in C_G(G)$ a class function on $H$ and $\beta \in C_G(G)$ one on $G$, one still has:

$$
\left( \text{Ind}_G^H \alpha | \beta \right)_G = \left( \alpha | \text{Res}_G^H \beta \right)_H.
$$

Proof. 

(i) By Proposition 9.1.2 (iv), there is an isomorphism of underlying vector spaces:

$$
\text{Hom}_{\mathbb{K}[H]}(W, \text{Res}^V_V) \cong \text{Hom}_{\mathbb{K}[G]}(\text{Ind}_W^G, V) \quad [\mathbb{K}-\text{Mod}].
$$

In particular, $\mathbb{K}$-linear dimensions match. Now dimensions of spaces of covariant morphisms were computed in Lemma 5.2.2 (ii) using the bilinear form, and this gives:

$$
\left( \chi_W | \chi_{\text{Res}^V_V} \right)_H = \left( \chi_{\text{Ind}_W^G} | \chi_{V} \right)_G.
$$

(ii) Notice that $\text{Res}_G^H : C_G(G) \to C_G(H)$ and $\text{Ind}_H^G : C_G(H) \to C_G(G)$ are linear maps. For $\mathbb{K}$-valued characters, (i) holds. Since characters of a finite group generate the space of class functions, the formula holds of all class functions. \( \square \)

9.4 Exercises

9.4.1. Exercise.

1. Show that $\text{Ind}_{(1)}^G \text{triv} \simeq \text{reg}_G$.

2. For finite $G$, deduce from Frobenius reciprocity that $\text{reg} = \sum_{\chi \in \text{Irr}(G)} \dim \chi \cdot \chi$.

9.4.2. Exercise. Let $G$ be a finite group and $K \leq H \leq G$ be two subgroups. Let $W$ be a representation of $K$. Prove that $\text{Ind}_K^G W \simeq \text{Ind}_K^H (\text{Ind}_H^G W) \quad [\mathbb{C}[G]-\text{Mod}]$.

9.4.3. Exercise. Let $H \leq G$ be finite groups and $\mathbb{K}$ be a field. Let $W$ be a representation of $H$ and $V = \text{Ind}_H^G W$. Let $\hat{\psi}_W$ coincide with $\psi_W$ on $H$ and equal $0$ on $G \setminus H$. Prove that:

$$
\chi_V(g) = \frac{1}{|H|} \sum_{x \in \text{Sym}(G)} \hat{\psi}_W(x^g).
$$

(\ast) 9.4.4. Exercise. Let $\text{perm} : \text{Sym}(4) \to \text{GL}_4(\mathbb{K})$ be the permutation representation. Now view $\text{Sym}(4)$ as a subgroup of $\text{Sym}(5)$. Show that $\text{perm}$ does not extend to $\text{Sym}(5) \to \text{GL}_4(\mathbb{K})$.

10 Frobenius complement theorem

Abstract. Frobenius' famous complement theorem deals with certain finite group-theoretic configurations. There is no known full proof avoiding character theory.
10.1 Frobenius pairs

10.1.1. Definition. A **Frobenius pair** is a pair of groups \((H < G)\) such that:

- \(H\) is self-normalising in \(G\), viz. \(N_G(H) = H\);
- \(H\) has trivial intersections with distinct conjugates, viz. \(G\) satisfies:
  \[
  (\forall g)[(H^g = H) \lor (H \cap H^g = \{1\})].
  \]

A subgroup with trivial intersections looks like this:

(It is harder to draw self-normalisation phenomena.) The conjunction is sometimes called *malnormality* of \(H\) in \(G\) but we cannot recommend the terminology.

10.1.2. Examples.

- Let \(k\) be a field, \(A = (K; +)\) be its additive group and \(M = (K^*, \cdot)\) be its multiplicative group. Then:
  \[
  A \rtimes M = \left\{ \begin{pmatrix} m & a \\ 0 & 1 \end{pmatrix} : (a, m) \in A \times M \right\},
  \]
  with \(M\) embedding to \(\text{diag}(1, m)\). Then \((M < A \rtimes M)\) is a Frobenius pair.

- While affine configurations are inspirational, they are not typical of Frobenius pairs. For example, there is a Frobenius pair with \(H \simeq \text{SL}_2(\mathbb{F}_3)\).

*Infinite* Frobenius pairs are a desperate topic.\(^8\)

10.1.3. Remarks.

- In the literature, \(G\) is sometimes called a *Frobenius group* and \(H\) its *Frobenius complement*.

  It can be proved (but it requires tools not available in this class) that if \((H_1 < G)\) and \((H_2 < G)\) are finite Frobenius pairs with the same \(G\), then there is \(g \in G\) with \(H_2^g = H_1\). So up to isomorphism there is *at most* one way in which a finite group can be the large group of a Frobenius pair, and the phrase ‘finite Frobenius group’ makes sense.

  This is completely not true with infinite groups.

- The above however supports the following question (Y. Tamer): is there a first-order formula characterising Frobenius groups among finite groups?

---

10.2 The geometry of a Frobenius pair

10.2.1. Notation. For \((H < G)\) a Frobenius pair, we let:
\[
N = \left( G \setminus \bigcup_{x \in G} H^x \right) \cup \{1\}.
\]

Despite suggestive notation, this is just a subset. It is normal (viz. closed under \(G\)-conjugation) and closed under \(^{-1}\). In general, no more can be said; for infinite groups, one could even have \(N = \{1\}\).

10.2.2. Lemma. Let \((H < G)\) be a Frobenius pair and \(g \in G\) with \(g \neq 1\). Then exactly one of the following two occurs:

- \(g \in \bigcup_{x \in G} H^x\), there is a unique \(c \in G/H\) such that \(gc = c\), and \(g^G \cap H\) is a single \(H\)-conjugacy class;
- \(g \in N\), there is no \(c \in G/H\) such that \(gc = c\), and \(g^G \cap H = \emptyset\).

**Proof.** If \(g \in H^x\), then \(gx^{-1}H = x^{-1}g^{-x}H = x^{-1}H\). If also \(gaH = aH\), then \(ga \in aH\) and \(g^a \in H\). So \(g \neq H^x \cap H^{a^{-1}}\), which forces \(xa \in N_G(H) = H\). Hence \(x^{-1}H = aH\) and the coset solution is unique. Suppose \(h_1, h_2 \in g^G \cap H\); they are not 1. There are \(x_i, x_j \in G\) with \(h_i = g^{x_i}\). In particular,
\[
h_1 = g^{x_i} = h_2^{x_i} x_i \in H \cap H^{x_i} x_i \setminus \{1\}.
\]
So \(x_i^{-1} x_j \in N_G(H) = H\), meaning that \(h_1\) and \(h_2\) are \(H\)-conjugate.

If \(g \in N\), then \(g^G \cap H = \emptyset\). If however \(gc = c\) for some coset \(c = aH\), then \(g \in H^{a^{-1}}\), a contradiction. So the equation \(gc = c\) has no solution in \(G/H\). \(\square\)

10.3 Frobenius’ complement theorem

The following remarkable result is due to Frobenius. We repeat that no fully character-free proof is known.

10.3.1. Theorem. Let \((H < G)\) be a finite Frobenius pair. Then there is a normal subgroup \(N \triangleright G\) such that \(G = N \rtimes H\).

**Proof.** There is no conflict with our notation since the only candidate is:
\[
N = \left( G \setminus \bigcup_{g \in G} H^g \right) \cup \{1\}.
\]
But there is no clear group-theoretic reason why \(N\) should be closed under product, and the proof requires a serious detour through character theory. We begin with a simple computation.

**Step 1.** \(#N = |G|/|H|\).

**Verification.** Subsets of the form \((H \setminus \{1\})^* = H^* \setminus \{1\}:

\[
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\]
• have $|H| - 1$ elements,
• are disjoint or equal,
• form a family parametrised by $G/N_G(H) = G/H$, with $|G|/|H|$ members.

So:

$$G \setminus N = \bigcup_{x \in G} H^x \setminus \{1\} = \bigcup_{x \in G} (H \setminus \{1\})^x$$

has exactly $|G|/|H| (|H| - 1) = |G| - |G|/|H|$ elements. Hence $N$ has exactly $|G|/|H|$ elements. ◆

The key step will be to use Frobenius’ reciprocity formula and prove that for a Frobenius pair, every irreducible character of $H$ extends to an irreducible character of $G$. (This is quite false in general; see exercise 9.4.4.)

The naive guess when trying to extend $\psi \in \text{Irr}_C(H)$ to $\psi \in \text{Irr}_C(G)$ would be $\text{Ind}_H^G \psi$. However, recall that 'Ind expands the dimension by a factor $[G : H]$, so $\text{Res}_H^G \text{Ind}_H^G \psi \neq \psi$. Indeed,

$$(\text{Res}_H^G \text{Ind}_H^G \psi)(1) = (\text{Ind}_H^G \psi)(1) = [G : H]\psi(1) = \psi(1).$$

The obstacle would disappear if $\psi(1)$ were 0. This suggests to consider the class function $\psi(h) - \psi(1)$, viz. $\psi - \psi(1)\text{triv}_H$.

**Step 2.** Let $\alpha: H \to \mathbb{C}$ be a class function with $\alpha(1) = 0$. Then $\text{Ind}_H^G \alpha$ is the unique class function on $G$ which • extends $\alpha$, and • vanishes on $N$.

**Verification.** Uniqueness is obvious since $G = N \cup \bigcup_{x \in G} H^x$.

By Lemma 10.2.2, if $g \in G$ is conjugate to two elements $h_1, h_2 \in H$, then $h_1$ and $h_2$ are already $H$-conjugate. In particular $\alpha(h_1) = \alpha(h_2)$ and we may define $\hat{\alpha}(g) = \alpha(h_1)$. On $N$ we let $\hat{\alpha}(x) = 0$, which is consistent since $N \cap H = \{1\}$ and $\alpha(1) = 0$. So there is an extension of $\alpha$ to a $G$-class function vanishing on $N$.

We must prove that $\hat{\alpha}$ so constructed actually equals:

$$(\text{Ind}_H^G \alpha)(g) = \sum_{x \in G/H: \ g^x \in \alpha} \alpha(g^x).$$

So let $g \in G$; we may assume $g \neq 1$. We use Lemma 10.2.2. If $g \in N$, the sum giving $\text{Ind}$ is empty, so $(\text{Ind} \alpha)(g) = 0 = \hat{\alpha}(g)$. If on the other hand $g \in H^x$, then the sum contains only term, namely $\alpha(g^x) = \hat{\alpha}(g^x) = \hat{\alpha}(g)$. ◆

**Step 3.** Every irreducible complex character $\psi \in \text{Irr}_C(H)$ extends to some $\psi \in \text{Irr}_C(G)$. Moreover we may suppose $N \subseteq \ker \psi$.

**Verification.** The trivial character triv$_H$ certainly extends to triv$_G$. So consider $\psi \in \text{Irr}_C(H) \setminus \{\text{triv}_H\}$; we seek to extend it to an irreducible character of $G$. It will be:

$$\hat{\psi} = \text{Ind}_H^G[\psi - \psi(1)\text{triv}_H] + \psi(1)\text{triv}_G.$$  

(Keep in mind at all times that although $\text{Ind}_H^G C(H) \to C(G)$ is linear, it does not take triv$_H$ to triv$_G$. So $\hat{\psi}$ does not equal $\text{Ind}_H^G \psi$.) Many details are required. Throughout,
we write Ind for $\text{Ind}_H^G$ and Res for $\text{Res}_H^G$.

First let $d = \psi(1) \in \mathbb{N}$. Now let $\alpha = \psi - d\text{triv}_H$, an $H$-class function with $\alpha(1) = 0$.

By Step 2, Ind $\alpha$ is a $G$-class function extending $\alpha$. Thus $\alpha = \text{Res} \text{Ind} \alpha$. As said, we let:

$$\hat{\psi} = \text{Ind} \alpha + d\text{triv}_G.$$ 

Then:

$$\text{Res} \hat{\psi} = \text{Res} \text{Ind} \alpha + d\text{triv}_H = \alpha + d\text{triv}_H = \psi.$$ 

But it remains to prove that $\hat{\psi}$ is a character of $G$.

First use linearity of Ind, giving:

$$\hat{\psi} = \text{Ind}(\psi - d\text{triv}_H) + d\text{triv}_G = \text{Ind}\psi - d \text{Ind} \text{triv}_H + d\text{triv}_G.$$ 

Now recall that Ind $\psi$ and Ind $\text{triv}_H$ are indeed characters of $G$ by Lemma 9.2.1. So $\hat{\psi}$ is a $\mathbb{Z}$-linear combination of characters of $G$, hence a $\mathbb{Z}$-linear combination of irreducible characters of $G$. We prove that there is only term by computing $\langle \hat{\psi} | \hat{\psi} \rangle$. This involves Frobenius reciprocity.

By reciprocity (extended to all class functions), bearing in mind Res $\text{triv}_G = \text{triv}_H$ and Res Ind $\alpha = \alpha$, one finds:

$$\langle \hat{\psi} | \hat{\psi} \rangle_G = \langle \text{Ind} \alpha + d\text{triv}_G, \text{Ind} \alpha + d\text{triv}_G \rangle_G$$
$$= \langle \text{Ind} \alpha, \text{Ind} \alpha \rangle_G + d \langle \text{Ind} \alpha, \text{triv}_G \rangle_G + d^2 \langle \text{triv}_G, \text{triv}_G \rangle_G$$
$$= (\alpha, \text{Res} \text{Ind} \alpha)_H + d (\text{Res} \text{triv}_G[\alpha]_H + d (\alpha \text{Res} \text{triv}_H)_H + d^2$$
$$= (\alpha + d\text{triv}_H[\alpha] + d\text{triv}_H)_H$$
$$= (\psi | \psi)_H$$
$$= 1.$$ 

Recall that $\hat{\psi}$ is a $\mathbb{Z}$-linear combination of irreducible characters of $G$, say $\hat{\psi} = \sum n_i \psi_i$, in obvious notation. Then by orthonormality, $\langle \hat{\psi} | \hat{\psi} \rangle_G = \sum n_i^2 = 1$. So $\hat{\psi}$ is itself either an irreducible character or the opposite of one. However $\hat{\psi}$ extends $\psi$, so $\hat{\psi}(1) = \psi(1) = d \in \mathbb{N}$ and $\psi \in \text{Irr}_C(G)$.

Last, for $g \in N$, the equation $gc = c$ has no solutions by Lemma 10.2.2. So any sum of the form

$$\langle \text{Ind} \beta | \gamma \rangle = \sum_{c \in G \cap H} \beta(g^c)$$

is actually empty. So there remains only $\hat{\psi}(g) = \psi(1) \text{triv}_G(g) = \psi(1) = \hat{\psi}(1)$, meaning $N \subseteq \ker \hat{\psi}$.

\[\square\]

**Step 4.** $N$ is a normal subgroup of $G$.

\[\square\]

**Verification.** For $\psi \in \text{Irr}_C(H)$, let $\hat{\psi} \in \text{Irr}_C(G)$ as in Step 3. Recall that $N \subseteq \ker \hat{\psi}$. Now let:

$$K = \bigcap_{\psi \in \text{Irr}_C(H)} \ker \hat{\psi},$$

a normal subgroup of $G$. We shall prove that $K = N$. The inclusion $N \subseteq K$ is by Step 3.

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We now show \( K \subseteq N \). This relies on proving \( K \cap H = \{1\} \). Indeed let \( h \in K \cap H \). Then for \( \psi \in \text{Irr}_C(H) \), one has \( \psi(h) = \psi(h) = \psi(1) = \psi(1) \). Hence \( h \in \bigcap_{\psi \in \text{Irr}_C(H)} \ker \psi = \{1\} \) by Theorem 7.1.4 (i). Since \( K \) is a normal subgroup, it avoids all conjugates of \( H \), viz. \( K \subseteq N \).

Hence \( N = K \) is a normal subgroup disjoint from \( H \). Now \(|N| = \#N = \frac{|G|}{|H|} \) by Step 1, which implies \( G = N \rtimes H \).

### 10.3.2. Remarks.

- Thompson proved that \( N \) must be nilpotent.\(^9\)
- If \( H \) has even order, \( N \) is even abelian. This is not true in general: see exercise 10.4.2.
- But \( H \) can be non-soluble: there is a finite Frobenius group with \( H \cong \text{SL}_3(\mathbb{F}_4) \).

### 10.3.3. Remark. Frobenius groups have a lovely application to Wedderburn's theorem (finite skew-fields are commutative) going through Desarguesian planes.\(^10\) Good publicity for my other \textit{NMK} lecture notes!

### 10.4 Exercises

#### 10.4.1. Exercise. Let \( G \) be a finite group acting transitively on a set \( X \). Suppose that every \( g \neq 1 \) fixes at most one element of \( X \). Prove that \( N = \{ \text{fixed-point free elements} \} \cup \{1\} \) is a normal subgroup.

#### 10.4.2. Exercise. Consider the group:

\[
N = \left\{ \begin{pmatrix} 1 & x & z \\ 1 & y & 1 \\ 1 \end{pmatrix} : (x, y, z) \in \mathbb{F}_3^3 \right\},
\]

and the map:

\[
\sigma \begin{pmatrix} 1 & x & z \\ 1 & y & 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x & 4z \\ 1 & 2y & 1 \\ 1 \end{pmatrix}.
\]

Prove that \( \sigma \) is an automorphism of order 3 of \( N \). Now prove that \( \langle (\sigma) \rangle \leq N \rtimes \langle (\sigma) \rangle \) is a Frobenius pair.

#### 10.4.3. Exercise. Let \( (H < G) \) be a finite Frobenius pair, say \( G = N \rtimes H \). Let \( \pi : G \to G/N \cong H \) be the quotient map. Let \( \chi \in \text{Irr}_C(G) \) be non-trivial. Prove that:

- either \( \chi = \psi \circ \pi \) for some \( \psi \in \text{Irr}_C(G/N) = \text{Irr}_C(H) \),
- or \( \chi = \text{Ind}_G^N \varphi \) for some \( \varphi \in \text{Irr}_C(N) \).

#### 10.4.4. Exercise (Bender's 'even' proof). This exercise contains no representation theory; on the contrary, it gives a character-free proof of Theorem 10.3.1 under extra assumptions.

Let \( (H < G) \) be a finite Frobenius pair. Suppose that \( H \) has even order.

---


1. Determine the cardinal of \( N = \bigcup_{g \in G} H^2 \cup \{1\} \).

2. Prove that if \( i, j \) are two involutions (of any group), then \((ij)^i = (ij)^{-1}\).

3. Let \( i \in H \) be an involution. Prove that for \( g \in G \setminus H \), one has \( 1 \neq ig^{-1}ig \in N \).

4. Determine the cardinal of \( R^* = \{ig^{-1}ig : g \in G \setminus H \} \subseteq N \).

5. Prove that \( N = R^* \cup \{1\} \).

6. Conclude that \( N \) is a subgroup of \( G \).

Notes.

- The method even proves abelianity of \( N \) under our assumption that \( H \) has even order. This is not true in general; however \( N \) is nilpotent (see Remarks 10.3.2).

- To date, there is no known full character-free proof of Theorem 10.3.1.¹

11 Real, purely complex, quaternionic representations

Abstract. §11.1 gives a correspondence between bilinear forms on \( V \) and morphisms \( V \cong V^* \). §11.2 discusses the irreducible case. Then §11.3 introduces real, purely complex, and quaternionic representations.

This section discusses only complex representations. The irreducible ones will be classified into real, complex, and quaternionic representations, in a technical sense. To avoid clash in terminology, we make an effort to talk about '\( \mathbb{C} \)-linear representations', and to refer to the second case as 'purely complex'.

11.1 Bilinear forms and duality

For \( V \) a \( \mathbb{K} \)-vector space, we let \( \text{Bil}_\mathbb{K}(V \times V, \mathbb{K}) \) be the space of \( \mathbb{K} \)-bilinear forms on \( V \). By definition, one has \( \text{Bil}_\mathbb{K}(V \times V, \mathbb{K}) \cong \text{Hom}_\mathbb{K}(V \otimes \mathbb{K} V, \mathbb{K}) \).

11.1.1. Definition. A bilinear form on a representation \( V \) is preserved by \( G \) if:

\[
(\forall g \in G)(\forall x \in V)(\forall y \in V)(\beta(gx, gy) = \beta(x, y)).
\]

We shall simply say that \( \beta \) is a bilinear \( G \)-form.

We let \( G\text{-Bil}_\mathbb{K}(V \times V, \mathbb{K}) \leq \text{Bil}_\mathbb{K}(V \times V, \mathbb{K}) \) be the subspace of bilinear \( G \)-forms. When no confusion can arise, we simply write \( G \)-Bil \( \leq \text{Bil} \).

11.1.2. Remark. We avoid writing \( \text{Bil}_{\mathbb{K}[G]}(V \times V, \mathbb{K}) \), because bilinear \( G \)-forms are not \( \mathbb{K}[G] \)-bilinear. Likewise, '\( G \)-bilinear' is slightly confusing.

11.1.3. Proposition. Let \( V \) be a finite-dimensional vector space.

(i) There is a natural isomorphism \( \text{Hom}_\mathbb{K}(V, V^*) \xrightarrow{\text{Hom}} \text{Bil}_\mathbb{K}(V \times V, \mathbb{K}) \mathbb{K} \mathbb{K} \text{-Mod} \). ¹

¹See however a very interesting entry on Terence Tao’s webpage, https://terrytao.wordpress.com/2013/06/24/a-fourier-analytic-proof-of-frobenius-theorem/
(ii) The above induces a natural bijection:

\[
\{ \text{linear isomorphisms } f : V \cong V^* [\mathbb{K}-\operatorname{Mod}] \} \\
\leftrightarrow \{ \text{non-degenerate bilinear forms } \beta : V \times V \to \mathbb{K} \}.
\]

(iii) If \( V \) is also a representation of a group \( G \), then (i) restricts to \( \operatorname{Hom}_{\mathbb{K}[G]}(V, V^*) \to G\text{-}\operatorname{Bil}(V \times V, \mathbb{K}) \), which like in (ii) induces a natural bijection:

\[
\{ \text{isomorphisms of representations } f : V \cong V^* [\mathbb{K}[G]-\operatorname{Mod}] \} \\
\leftrightarrow \{ \text{non-degenerate bilinear } G\text{-forms } \beta : V \times V \to \mathbb{K} \}.
\]

11.1.4. Remark. In (iii), the isomorphism \( \operatorname{Hom}_{\mathbb{K}[G]}(V, V^*) \approx G\text{-}\operatorname{Bil} \) can be considered in \( \mathbb{K}[G]-\operatorname{Mod} \) or in \( \mathbb{K}-\operatorname{Mod} \) without loss of information, since the natural action of \( G \) on each side is trivial.

\[\textbf{Proof.}\]

(i) Truly, in abstract terms, this is because in \( \mathbb{K}-\operatorname{Mod} \) one has isomorphisms:

\[
\operatorname{Bil}_{\mathbb{K}}(V \times V, \mathbb{K}) \cong \operatorname{Hom}(V \otimes V, \mathbb{K}) \cong (V \otimes V)^* \cong V^* \otimes V^* \\
\cong \operatorname{Hom}(V^{**}, V^*) \cong \operatorname{Hom}(V, V^*) [\mathbb{K}-\operatorname{Mod}].
\]

One may prefer a casual approach. To \( \mathbb{K}\text{-linear } f : V \to V^* \), associate the map

\[\beta_f(x, y) = f(x)(y) \in \mathbb{K} \text{.} \]

Clearly \( \beta_f \) is bilinear. Moreover \( f \mapsto \beta_f \) is linear.

Conversely, to bilinear \( \beta : V \times V \to \mathbb{K} \), associate \( \beta_f(x) = \beta(x, \cdot) \in V^* \). Then \( f_{\beta} : V \to V^* \) is linear. Moreover \( \beta \mapsto f_{\beta} \) is linear. These constructions are inverses of each other, hence linear isomorphisms.

(ii) Work in the notation above. Suppose \( f : V \cong V^* \) is a linear isomorphism. If \( x \in V \) is such that \( \beta_f(x, \cdot) = 0 \), then \( f(x) = 0 \) and therefore \( x = 0 \). So \( \beta_f \) is left-non-degenerate; by finite-dimensionality, this is enough.

Conversely suppose that \( \beta : V \times V \to \mathbb{K} \) is a non-degenerate bilinear form. If \( f_{\beta}(x) = 0 \), then \( x = 0 \) by non-degeneracy. So \( f_{\beta} : V \cong V^* [\mathbb{K}-\operatorname{Mod}] \).

(iii) Return to the construction, in the same notation.

Suppose that \( f \) is \( G\text{-covariant}. \) Thus \( f(g \cdot x) = g \cdot f(x) \). Let \( \varphi = f(x) \in V^* \), so that \( f(gx) = g \cdot \varphi \). By definition of the dual representation, for \( y \in V \) one has \( (g \cdot \varphi)(y) = \varphi(g^{-1}y) \). Therefore \( f(gx)(y) = f(x)(g^{-1}y) \) and finally:

\[\beta_f(gx, gy) = f(gx)(gy) = f(x)(g^{-1}gy) = f(x)(y) = \beta_f(x, y),\]

so \( \beta_f \) is preserved by \( G \).

Conversely, if \( \beta \) is preserved by \( G \), then:

\[f_{\beta}(g \cdot x)(y) = \beta(gx, y) = \beta(x, g^{-1}y) = f(x)(g^{-1}(y)) = (g \cdot f(x))(y),\]

so \( f_{\beta}(g) = g \cdot f(x) \) inside \( V^* \).

\[\Box\]
11.2 Real, purely complex, quaternionic representations

With Proposition 11.1.3, one could expect the theory to divide in two: either \( V \cong V^* \) or not, based on existence or not of non-degenerate bilinear \( G \)-forms. But the theory of bilinear forms divides itself into two main subtopics, so there are three cases in total.

11.2.1. Definition. A bilinear form \( \beta \) is:

- symmetric if \((\forall x)(\forall y)(\beta(y, x) = \beta(x, y))\);
- alternating if \((\forall x)(\forall y)(\beta(y, x) = -\beta(x, y))\).

We write \( \text{Bil}' \), resp. \( \text{Bil}^a \) for symmetric, resp. alternating bilinear forms. \( G\)-\( \text{Bil}' \) and \( G\)-\( \text{Bil}^a \) are defined likewise.

One also says skew-symmetric for alternating.

11.2.2. Proposition. Let \( G \) be a finite group; work over \( \mathbb{C} \). Let \( V \) be an irreducible, \( \mathbb{C} \)-linear representation. Then:

(i) every non-zero, bilinear \( G \)-form on \( V \) is non-degenerate;

(ii) exactly one of the following three case occurs:
- \([\text{real}]\): \( G\)-\( \text{Bil}' \) has dimension 1 and \( G\)-\( \text{Bil}^a = \{0\} \).
- \([\text{purely complex}]\): \( G\)-\( \text{Bil}' = G\)-\( \text{Bil}^a = \{0\} \).
- \([\text{quaternionic}]\): \( G\)-\( \text{Bil}' \) has dimension 1 and \( G\)-\( \text{Bil}^a = \{0\} \).

**Proof.** The claims follow from two lemmas.

11.2.3. Lemma. Let \( V \) be an irreducible \( \mathbb{C} \)-linear representation of a finite group \( G \). Then every non-zero, bilinear \( G \)-form is non-degenerate. Moreover \( \dim G\)-\( \text{Bil} = 0 \) or 1.

**Proof.** Recall from Proposition 11.1.3 that we have a 'dictionary': \( G\)-\( \text{Bil}_C( V \times V, \mathbb{C}) \cong \text{Hom}_C[G](V, V^*) \). Now \( V \) is irreducible, and therefore so is \( V^* \). Thus by Schur's Lemma, \( \text{Hom}_C[G](V, V^*) \) is either trivial or 1-dimensional.

Let \( \beta: V \times V \to \mathbb{C} \) be a bilinear \( G \)-form. Let \( f_\beta: V \to V^* \) [\( \mathbb{C}\text{-Mod} \)] be given by Proposition 11.1.3; then \( f_\beta \neq 0 \). By Schur's Lemma again, \( f_\beta \) is an isomorphism. Translating back through Proposition 11.1.3, \( \beta_f \) is non-degenerate.

11.2.4. Lemma. Let \( V \) be any representation over a field of characteristic \( \neq 2 \). Then \( \text{Bil} = \text{Bil}' \oplus \text{Bil}^a \), and \( G\)-\( \text{Bil} = G\)-\( \text{Bil}' \oplus G\)-\( \text{Bil}^a \).

**Proof.** Let \( \beta: V \times V \to \mathbb{C} \) be bilinear. Notice how:

\[
\beta'(x, y) = \frac{\beta(x, y) + \beta(y, x)}{2} \quad \text{and} \quad \beta^a(x, y) = \frac{\beta(x, y) - \beta(y, x)}{2}
\]

are two bilinear forms; the first is symmetric and the second is alternating. Moreover \( \beta = \beta' + \beta^a \). Last, if \( \beta \) is a \( G \)-form, so are \( \beta' \) and \( \beta^a \).

We prove the Proposition. The first lemma implies (i) so we move to (ii). The three cases are mutually exclusive. If there is no non-trivial bilinear \( G \)-form, we are in
the purely complex case. So suppose there is one, $\beta \neq 0$; decompose it as $\beta = \beta^s + \beta^a$. By the first lemma again, $\dim G\text{-Bil} = 1$, so $G\text{-Bil} = \langle \beta \rangle$. In particular there is $\lambda \in \mathbb{C}$ with $\beta^s = \lambda \beta$. If $\lambda \neq 0$ then $\beta^s \neq 0$; thus $\langle \beta^s \rangle = G\text{-Bil}^s = G\text{-Bil}$ and there remains $G\text{-Bil}^a = \langle 0 \rangle$: this is the real case. Otherwise $\lambda = 0$, meaning $\beta = \beta^a \neq 0$: we reach the quaternionic case.

11.2.5. Remarks.

- It is enough to have $\mathbb{K}$ a good field of characteristic $\neq 2$.
- Beware of terminology (1). The character $\chi_V$ is real-valued iff $V$ is (real or quaternionic). The proof is immediate: $\chi_V$ is real-valued iff $\chi_{V^*} = \chi_V^* = \overline{\chi_V} = \chi_V$ iff $V^* \cong V [\mathbb{C}[G]\text{-Mod}]$, and we apply Proposition 11.1.3.
- Beware of terminology (2). In all three cases, $\text{End}_{\mathbb{K}[G]}(V) = \mathbb{C} \text{Id}_V$ by irreducibility, so quaternions will never emerge as Schur’s field. A better name for the third case could have been Weyl’s neologism symplectic, since a symplectic form is a non-degenerate, alternating bilinear form.
- Reason for the terminology is in exercise 11.4.3.

11.3 More on real and quaternionic geometries

We elaborate on the cases delineated by Proposition 11.2.2. Recall that a map $f: V_1 \to V_2$ between complex vector spaces is semi-linear if it is additive but $f(\lambda v) = \lambda f(v)$. (It is then $\mathbb{R}$-linear but ‘twists’ the action of $i$.)

11.3.1. Proposition. Let $G$ be a finite group; work over $\mathbb{C}$. Let $V$ be a $\mathbb{C}$-linear, irreducible representation. Then the following are equivalent:

(i) there exists a non-degenerate, symmetric, bilinear $G$-form on $V$;

(ii) there exists a $G$-covariant, semi-linear isomorphism $\sigma: V \to V$ with $\sigma^2 = \text{Id}_V$;

(iii) there exists a real, $G$-invariant vector space $W \subseteq V$ with $V = W \oplus iW$.

11.3.2. Remarks. Let us rephrase (iii).

- In matrix form: there is a basis $\mathcal{B}$ in which all matrices $\text{Mat}_\mathcal{B} \rho(g)$ have real coefficients.
- In dimension-theoretic form: there is a real, $G$-invariant vector space $W \subseteq V$ with $\dim_\mathbb{R} W = \dim_\mathbb{C} V$ and $V = \langle W \rangle_\mathbb{C}$.
- In abstract form: there is an $\mathbb{R}$-linear representation $W'$ such that $V \cong W' \otimes_\mathbb{R} \mathbb{C}$, where $\mathbb{C}$ is equipped with the trivial $G$-action and the usual $\mathbb{R}$-action.

This accounts for the name: a $\mathbb{C}$-linear representation is real if it comes from an $\mathbb{R}$-linear representation.

11.3.3. Proof.

(i)$\Rightarrow$(ii). By Proposition 11.1.3, such a bilinear form induces an isomorphism $f_\beta: V \cong V^* [\mathbb{K}[G]\text{-Mod}]$. (We have not used symmetry of $\beta$ so far.)

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Let $\langle \cdot, \cdot \rangle$ a complex scalar product on $V$. It is the same as a semi-linear isomorphism $s: V \cong V^*$. If we average $\langle \cdot, \cdot \rangle$ using $G$, we may even take a complex scalar product $\langle \cdot, \cdot \rangle$ preserved by $G$. (This method was already in exercise 3.4.5.) Hence we may suppose that there is a semilinear, $G$-covariant isomorphism $s: V \cong V^*$. It satisfies $s(x)(y) = [x|y]$. In particular, for $\varphi \in V^*$, one has:

$$\langle s^{-1}\varphi | y \rangle = \varphi(y).$$

Finally let $\sigma = s^{-1} \circ f_\beta: V \to V$. It is a semi-linear, $G$-covariant isomorphism. So $\sigma^2: V \to V$ is a linear, $G$-covariant isomorphism. By irreducibility and Schur's Lemma, there is $\lambda \in \mathbb{C}$ with $\sigma^2 = \lambda \text{Id}_V$. Now:

$$\langle \sigma x | y \rangle = \|s^{-1}f_\beta(x)\|y\rangle = f_\beta(x)(y) = \beta(x, y),$$

so finally using symmetry of $\beta$:

$$\|\sigma x | y \rangle = \beta(x, y) = \beta(y, x) = \|\sigma y | x \rangle .$$

We apply this to $y = \sigma x$, getting:

$$\|\sigma x | \sigma x \rangle = \|\sigma \sigma x | x \rangle = [\lambda x | x \rangle = \lambda [x | x \rangle .$$

But for $x \neq 0$, both $\|\sigma x | \sigma x \rangle$ and $[x | x \rangle$ are positive real numbers; thus so are $\lambda$ and $\bar{\lambda}$. Up to rescaling by $\frac{1}{\sqrt{\lambda}}$, we may thus suppose $\lambda = 1$.

$(\text{iii}) \Rightarrow (\text{iii})$. Treat $V$ as a real vector space. There the linear map $\sigma$ satisfies $\sigma^2 = \text{Id}_V$, so it is diagonalisable with eigenvalues $\pm 1$. Let $W_+ = E_+(\sigma)$ and $W_- = E_{-\lambda}(\sigma)$. Then $V = W_+ \oplus W_-$. Moreover, each is clearly $G$-invariant.

We claim that $W_- = iW_+$. Indeed, if $w_+ \in W_+$, then $\sigma(iw_+) = -i\sigma(w_+) = -iw_-$, so $iw_+ \in W_-$. The converse inclusion is proved similarly. It follows that $W_+$ and $W_-$ have the same real dimension.

Thus $V = W_+ \oplus iW_-$, and in particular, $(W) = V$ as a $\mathbb{C}$-vector space. Moreover, $\dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} V = \frac{1}{2}(\dim_{\mathbb{R}} W_+ + \dim_{\mathbb{R}}(iW_+)) = \dim_{\mathbb{R}} W_+$.

$(\text{iii}) \Rightarrow (\text{i})$. Let $W \subseteq V$ be a real, $G$-invariant, subspace such that $V = W \oplus iW$; hence every $v \in V$ writes uniquely as $v = x + iy$ with $x, y \in W$. Take a real scalar product on $W$. Averaging it, we may take a real scalar product $\langle \cdot, \cdot \rangle$ on $W$ which is preserved by $G$. Now simply put:

$$(x + iy|x' + iy') = (\langle x|x' \rangle - \langle y|y' \rangle) + (\langle x|y' \rangle + \langle y|x' \rangle) i \in \mathbb{C}.$$ 

Then $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-bilinear and $G$-covariant. Since it is non-zero, it is non-degenerate by Proposition 11.2.2.

In a very similar way, one proves the following.

**11.3.3. Proposition.** Let $G$ be a finite group and $V$ be a $\mathbb{C}$-linear, irreducible representation. Then the following are equivalent:

(i) there exists a non-degenerate, alternating, bilinear $G$-form on $V$;

(ii) there exists a $G$-covariant, semi-linear isomorphism $\sigma: V \to V$ with $\sigma^2 = -\text{Id}_V$. 

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11.4 Exercises

11.4.1. Exercise. Let $V$ be a $K$-linear representation of $G$. Define an action of $G$ on $\text{Bil}_K(V \times V, K)$ such that the map $\beta \mapsto f_\beta$ of Proposition 11.1.3 is a morphism of representations. Why could you expect inversions?

11.4.2. Exercise. Prove Proposition 11.3.3.

11.4.3. Exercise. Let $\mathbb{H}_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$ be the group of basic quaternions.

1. Compute the only irreducible, 2-dimensional character over $\mathbb{C}$, say $\chi_2$.

2. Prove that $\chi_4$ is not real. [Hint: extend the candidate group morphism $\mathbb{H}_8 \to \text{GL}_2(\mathbb{R})$ to the associative algebra of quaternions $\mathbb{H}$.]

3. Consider the following realisation of quaternions:\footnote{12}

$$\mathbb{H} = \left\{ \left( \begin{array}{c} z_1 \\ z_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{array} \right) : (z_1, z_2) \in \mathbb{C}^2 \right\} \leq M_4(\mathbb{C}) \quad [\mathbb{R}\text{-Alg}].$$

Realise $\chi_4$ by giving explicit $\mathbb{H}_8 \to \mathbb{H}^* \leq \text{GL}_2(\mathbb{C})$.

11.4.4. Exercise (real-valued characters, and groups of odd order). Let $G$ be a finite group. We work over $\mathbb{C}$. A (complex) character $\chi$ is real-valued if $\chi = \overline{\chi}$. A conjugacy class $\gamma \in \text{Conj}(G)$ is real if $\gamma^{-1} = \gamma$.

1. Prove that $\gamma$ is real iff $\chi \in \text{Irr}_\mathbb{C}(G)$ (where $\chi(\gamma) \in \mathbb{R}$). [Hint: column orthogonality.]

2. Let $d = \# \text{Irr}_\mathbb{C}(G) = \# \text{Conj}(G)$. Let $\sigma_1, \sigma_2 \in \text{Sym}(d)$ be given by $\overline{\chi_i} = \chi_{\sigma_i(i)}$ and $\gamma_{\sigma_i} = \gamma_{\sigma_i(i)}$.

Also let $\rho = \text{perm} \text{Sym}(d) : \text{Sym}(d) \to \text{GL}_d(\mathbb{C})$ be the permutation representation. Finally let $M \in M_d(\mathbb{C})$ be the character table.

What are $\rho(\sigma_1)M$ and $M\rho(\sigma_2)$?

3. Deduce that the number of real conjugacy classes equals the number of real-valued characters. [Hint: what is $\text{tr} \rho(\sigma_1)M$?]

4. Deduce that $|G|$ is odd iff the only real-valued irreducible character of $G$ is trivial. [This question contains no representation theory.]

5. Deduce that if $|G|$ is odd, then $\# \text{Conj}(G) \equiv |G| [16]$.

12 The Frobenius-Schur formula

Abstract. The Frobenius-Schur formula determines whether an irreducible, $\mathbb{C}$-linear representation is real, purely complex, or quaternionic in the sense of § 11.
12.1 Statement
Recall that an irreducible, \(C\)-linear representation \((V, \rho)\) is:

- **real** if there exists a non-zero, symmetric, bilinear \(G\)-form on \(V\);
- **purely complex** if there exists no non-zero, bilinear \(G\)-form on \(V\);
- **quaternionic** if there exists a non-zero, alternating, bilinear \(G\)-form on \(V\).

12.1.1 Theorem. Let \(G\) be a finite group and \(V\) be a \(C\)-linear, irreducible representation with character \(\chi\). Then:

\[
\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \begin{cases} 
1 & \text{if } V \text{ is real} \\
0 & \text{if } V \text{ is purely complex} \\
-1 & \text{if } V \text{ is quaternionic.}
\end{cases}
\]

There are no other values.

The proof requires a short geometric digression.

12.2 Sym and Alt

Here Sym and Alt do not stand for symmetric and alternating groups, but for certain factors of tensor powers. These construction are natural in geometry.

12.2.1 Definition. Let \(V\) be a vector space and \(k\) be an integer. Let \(\Sigma_k\) be the symmetric group on \(\{1, \ldots, k\}\).

- The \(k\)th symmetric power of \(V\) is:

\[
\text{Sym}^k(V) = \left( \bigotimes^k V \right) / \left\{ (v_1, \ldots, v_k) \in V^k, \sigma \in \Sigma_k \right\}
\]

- The \(k\)th exterior power of \(V\) is:

\[
\text{Alt}^k(V) = \left( \bigotimes^k V \right) / \left\{ (v_1, \ldots, v_k) \in V^k, \sigma \in \Sigma_k \right\}
\]

12.2.2 Notation. In the notation above, we let:

- \(v_1 \cdots v_k\) be the image of \(v_1 \otimes \cdots \otimes v_k\) in \(\text{Sym}^k(V)\);
- \(v_1 \wedge \cdots \wedge v_k\) be the image of \(v_1 \otimes \cdots \otimes v_k\) in \(\text{Alt}^k(V)\).

12.2.3 Lemma. Let \(G\) be a group and \(\mathbb{K}\) be a field of characteristic \(\neq 2\). Let \(V\) be a \(\mathbb{K}\)-linear, finite-dimensional representation of \(G\). Then:

(i) \(\text{Sym}^2(V)\) and \(\text{Alt}^2(V)\) are naturally representations, under \(g \cdot (xy) = (gx)(gy)\) and \(g \cdot (x \wedge y) = (gx) \wedge (gy)\), extended linearly.

(ii) \(\text{Sym}^2(V)\) and \(\text{Alt}^2(V)\) are isomorphic to subrepresentations of \(V \otimes V\); moreover

\[
V \otimes V \simeq \text{Sym}^2(V) \oplus \text{Alt}^2(V) \quad \left[ \mathbb{K}[G] - \text{Mod} \right];
\]
(iii) \( \chi_{\text{Sym}^r(V)}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)) \) and \( \chi_{\text{Alt}^r(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)) \).

**Proof.**

(i) Clear.

(ii) Let \( B = \{ e_1, \ldots, e_n \} \) be a basis of \( V \). Then \( \{ e_i \otimes e_j : 1 \leq i, j \leq n \} \) is a basis of \( V \otimes V \).

- Let \( \Sigma_2 = \{ (12) \} \) act on \( V \otimes V \) by letting \( (12)(e_i \otimes e_j) = e_j \otimes e_i \), extended linearly. Let \( \sigma \) be the image of \( (12) \) in \( \text{End}_G(V \otimes V) \). Thus,
  \[
  \sigma(\sum \lambda_{i,j} e_i \otimes e_j) = \sum \lambda_{i,j} e_j \otimes e_i = \sum \lambda_{j,i} e_i \otimes e_j.
  \]

  In particular, \( \sigma(v_i \otimes v_j) = v_j \otimes v_i \) for all \( v_i, v_j \in V \).

- Notice \( G \)-covariance of \( \sigma \).

- Clearly \( \sigma^2 = \text{Id} \) in \( \text{End}_G(V \otimes V) \), so \( (\sigma - \text{Id})(\sigma + \text{Id}) = 0 \). Basic linear algebra implies \( \ker(\sigma - \text{Id}) = \text{im}(\sigma + \text{Id}) \) while \( \ker(\sigma + \text{Id}) = \text{im}(\sigma - \text{Id}) \), and also \( V \otimes V = \ker(\sigma - \text{Id}) \oplus \ker(\sigma + \text{Id}) \).

- Consider the following elements of \( V \otimes V \):

  \[
  s_{i,j} = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) \quad \text{for} \quad i \leq j \quad \text{and} \quad a_{i,j} = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i) \quad \text{for} \quad i < j.
  \]

  Let \( B_s = \{ s_{i,j} : i \leq j \} \) and \( B_a = \{ a_{i,j} : i < j \} \). Since \( \#(B_s \cup B_a) = n^2 \) and \( e_j \otimes e_j \in (B_s \cup B_a) \), we get that \( B_s \cup B_a \) is a basis.

- Now \( \langle B_s \rangle = \ker(\sigma - \text{Id}) \) and \( \langle B_a \rangle = \ker(\sigma + \text{Id}) \) follow easily.

- Last, \( \text{Sym}^r(V) = (V \otimes V)/\ker(\sigma - 1) \approx \ker(\sigma + 1) \). Moreover, \( s_{i,j} \) maps to \( e_i e_j \). Similarly, \( \text{Alt}^r(V) = (V \otimes V)/\ker(\sigma + 1) \approx \ker(\sigma + 1) \), and \( a_{i,j} \) maps to \( e_i \wedge e_j \).

- By \( G \)-covariance, all the above holds as representations of \( G \).

A byproduct of the proof is that \( \{ e_i e_j : i \leq j \} \) is a basis of \( \text{Sym}^r(V) \) and \( \{ e_i \wedge e_j : i < j \} \) is a basis of \( \text{Alt}^r(V) \).

(iii) Let \( g \in G \) be fixed. One could have started with an eigenbasis \( B = \{ e_1, \ldots, e_n \} \) under the action of \( g \), viz. \( g e_i = \lambda_i e_i \). Then notice that each \( e_i \otimes e_j \in V \otimes V \), each \( e_i e_j \in \text{Sym}^r(V) \), each \( e_i \wedge e_j \in \text{Alt}^r(V) \) is an eigenvector for \( g \) with eigenvalue \( \lambda_i \lambda_j \). Since vectors of this type form eigenbases of the corresponding spaces, we conclude that:

\[
\chi_{\text{Sym}^r(V)} = \sum_{i \leq j} \lambda_i \lambda_j \quad \text{and} \quad \chi_{\text{Alt}^r(V)} = \sum_{i < j} \lambda_i \lambda_j.
\]

On the other hand,

\[
\chi_V(g)^2 = \left( \sum_{i} \lambda_i \right)^2 = 2 \sum_{i < j} \lambda_i \lambda_j + \sum_{i} \lambda_i^2 = 2 \chi_{\text{Alt}^r(V)}(g) + \chi_V(g^2).
\]

This gives the desired formulas. \( \square \)
In practice one may want to remember that \( \{ e_i e_j : i \leq j \} \) is a basis of \( \text{Sym}^3(V) \) and 
\( \{ e_i \wedge e_j : i < j \} \) one of \( \text{Alt}^3(V) \).

12.3 Proof of the Frobenius-Schur formula

**Proof.** Let \( V \) be a \( \mathbb{C} \)-linear, irreducible representation of \( G \). We study \( G \)-covariant bilinear forms on \( V \) and the representation \( W = \text{Bil}_G(V \times V, \mathbb{C}) \). (See exercise 11.4.1.) Also let:

\[
W_\beta = \text{Bil}_G^\beta(V \times V, \mathbb{C}) = \{ \beta \in W : \beta \text{ is symmetric} \},
\]

and define \( W_a \) likewise. Finally let \( d_s = \dim W_s \) and \( d_a = \dim W_a \). By Proposition 11.2.2, notice that:

- \( V \) is real iff \( d_s = 1 \) and \( d_a = 0 \);
- \( V \) is purely complex iff \( d_s = d_a = 0 \);
- \( V \) is quaternionic iff \( d_s = 0 \) and \( d_a = 1 \).

So the integer \( d_s - d_a \in \{ 1, 0, -1 \} \) indicates the geometric type of \( V \). We must therefore prove that the formula computes \( d_s - d_a \).

By definition of the tensor product and the dual space,

\[
W = \text{Hom}_G(V \otimes V, \mathbb{C}) \simeq (V \otimes V)^* \simeq (V^* \otimes V^*) \quad [\mathbb{K}[G] \cdot \text{-Mod}].
\]

It is not hard to see that this induces isomorphisms:

\[
W_s = \text{Sym}^3(V^*) \quad [\mathbb{K}[G] \cdot \text{-Mod}] \quad \text{and} \quad W_a = \text{Alt}^3(V^*) \quad [\mathbb{K}[G] \cdot \text{-Mod}].
\]

By Lemma 12.2.3, their characters are:

\[
\chi_W(g) = \frac{1}{2} (\chi_{V^*}(g)^2 + \chi_{V^*}(g^3)) = \frac{1}{2} (\chi_{V^*}(g)^2 + \chi_{V^*}(g^3)),
\]

and

\[
\chi_W(g) = \frac{1}{2} (\chi_{V^*}(g)^2 - \chi_{V^*}(g^3)) = \frac{1}{2} (\chi_{V^*}(g)^2 - \chi_{V^*}(g^3)).
\]

Now the space of symmetric, bilinear \( G \)-forms is:

\[
G \text{-Bil}_G(V \times V, \mathbb{C}) = C_{W_s}(G) \simeq C_{\text{Sym}^3(V^*)}(G).
\]

Lemma 5.2.2 gave a formula for its dimension:

\[
d_s = \dim G \text{-Bil}_G^\beta(V \times V, \mathbb{C}) = \dim C_{W_s}(G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{2} (\chi_{V^*}(g)^2 + \chi_{V^*}(g^3)),
\]

and likewise:

\[
d_a = \dim G \text{-Bil}_G^\beta(V \times V, \mathbb{C}) = \dim C_{W_a}(G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{2} (\chi_{V^*}(g)^2 - \chi_{V^*}(g^3)).
\]

Therefore:

\[
d_s - d_a = \frac{1}{|G|} \sum_{g \in G} \chi_{V^*}(g^3).
\]
This looks like the desired formula, but there is a residual $\ast$. Fortunately $d_s$ and $d_a$ are integers, so finally:

$$\frac{1}{|G|} \sum_G \chi_V(g^2) = \frac{1}{|G|} \sum_G \overline{\chi_V}(g^2) = d_s - d_a = d_s - d_a.$$

\[\square\]

### 12.4 Exercises

**12.4.1. Exercise.** Return to the decomposition $\beta = \beta_s + \beta_a$ of Lemma 11.2.4. Prove that it is compatible with expressing $V \otimes V \cong \text{Sym}^2(V) \oplus \text{Alt}^2(V)$ (viz. draw a suitable commutative diagram).

**12.4.2. Exercise.**

1. Show that $\text{Sym}^k V$ enjoys the following universal property:

   any $k$-linear, symmetric map from $V^k$ to another vector space factor uniquely through $\text{Sym}^k V$.

2. Find a similar universal property describing $\text{Alt}^k V$.

**12.4.3. Exercise.**

1. Suppose $\mathcal{B}$ is a finite basis of $V$. Give bases of $\text{Sym}^k(V)$ and $\text{Alt}^k(V)$.

2. Compute characters.

(*) 3. Suppose that $K$ has characteristic coprime to $k!$. Find subrepresentations of $\otimes^k V$ isomorphic to $\text{Sym}^k(V)$, resp. $\text{Alt}^k(V)$.

**12.4.4. Exercise.** The purpose of this exercise is to construct the character table over $\mathbb{C}$ of $G = \text{Sym}(5)$. Let $\chi_4 = \text{perm} - \text{triv}$, which is irreducible.

1. Prove that $\text{Alt}^2 \chi_4$ is irreducible and has dimension 6.

2. Prove that $\text{Sym}^3 \chi_4$ is the sum of three irreducible characters.

3. Find an irreducible character of dimension 5.

4. Complete the character table of $\text{Sym}(5)$.

**12.4.5. Exercise.** Return to the character tables of $\text{Alt}(4), \text{Sym}(4), \text{Alt}(5), \text{Sym}(5)$. Find out which representations are real, purely complex, quaternionic.

### 13 The group algebra

**Abstract.** This section recasts representation theory in *module theory* (§13.1). The *group algebra* of $G$ over $K$ (§13.2) is precisely the associative $K$-algebra encoding the representation theory of $G$. We then translate orthogonality relations in terms of *central idempotents* (§13.3).
13.1 Modules, submodules, morphisms

13.1.1. Definition. Let $R$ be a ring. An $R$-module is a abelian group $M$ equipped with an action of $R$ on $M$ such that, for all $r, s \in R$ and $m, n \in M$:

- $(r + s) \cdot m = r \cdot m + s \cdot m$;
- $1 \cdot m = m$;
- $(r \cdot s) \cdot m = r \cdot (s \cdot m)$;
- $r \cdot (m + n) = r \cdot m + r \cdot n$.

(These do imply $0 \cdot m = 0$.) With the notion of a module comes that of a submodule.

13.1.2. Examples.
- The $\mathbb{Z}$-modules are the abelian groups. Then $\mathbb{Z}$-submodules are subgroups.
- If $K$ is a field, the $K$-modules are the $K$-vector spaces. Then $K$-submodules are $K$-linear subspaces.
- If $K$ is a field, the $K[X]$-modules are the $K$-vector spaces equipped with one distinguished linear endomorphism $f$. Then $K[X]$-submodules are $f$-invariant subspaces.

13.1.3. Definition. Let $R$ be a ring. An $R$-module $M$ is simple, or irreducible if the only two $R$-submodules of $M$ are $\{0\}$ and $M$.

We now introduce the suitable notion of morphism.

13.1.4. Definition. Let $R$ be a ring and $M_1, M_2$ two $R$-modules.
- An $R$-morphism is a map $\varphi : M_1 \to M_2$ such that for all $r \in R$ and $m, m' \in M_1$:
  $$\varphi(r \cdot m + m') = r \cdot \varphi(m) + \varphi(m').$$

One sometimes says that $\varphi$ is $R$-covariant.

- Let $R$ be a ring and $M_1, M_2$ be two $R$-modules. We let $\text{Hom}_R(M_1, M_2)$ stand for the collection of $R$-morphisms from $M_1$ to $M_2$. (A better notation would be $(M_1 \to M_2; \text{R-Mod})$.)

  If $R$ is clear from context, one simply writes $\text{Hom}(M_1, M_2)$.

- An $R$-isomorphism is a bijective $R$-morphism. We write $M_1 \cong M_2$ [$\text{R-Mod}$] (isomorphism in the category of $R$-modules).

Notice that $\text{Hom}_R(M_1, M_2)$ itself is an abelian group. However, in general it does not bear a natural $R$-module structure.

13.1.5. Examples.
- A $\mathbb{Z}$-morphism between $\mathbb{Z}$-modules is a group morphism between abelian groups.
- A $K$-morphism between $K$-vector spaces is a $K$-linear map.
- A $K[X]$-morphism between $K[X]$-modules $(V_1, f_1)$ and $(V_2, f_2)$ is a $K$-linear map $\varphi : f_1 \to f_2$ such that $\varphi \circ f_1 = f_2 \circ \varphi$.

We do not introduce tensor products over arbitrary rings. This would only create confusion since in this course, we only tensor over $K$, never over the group algebra.
13.2 The group algebra and representations

13.2.1. Definition. Let \( \mathbb{K} \) be a field and \( G \) be a group. The group algebra of \( G \) over \( \mathbb{K} \) is the following associative \( \mathbb{K} \)-algebra:

- let \( \mathbb{K}[G] \) be the vector space with basis \( \{ g : g \in G \} \);
- define multiplication on \( \mathbb{K}[G] \) by extending \( \mathbb{K} \)-linearly multiplication on \( G \).

Thus \( \dim_{\mathbb{K}} \mathbb{K}[G] = |G| \). The identity element is 1. The group algebra is associative because \( G \) is, but it is commutative iff \( G \) is.

13.2.2. Remark. Analysts like to think of \( \mathbb{K}[G] \) as the algebra of all functions \( G \rightarrow \mathbb{K} \). It is then equipped with the convolution product:

\[(f_1 \ast f_2)(g) = \sum_{x,y \in G, x+y = g} f_1(x)f_2(y).\]

One should check at once that \((\mathbb{K}[G]; +, \ast)\) and \((\mathbb{K}^G; +, \ast)\) are naturally isomorphic as associative \( \mathbb{K} \)-algebras: just take \( \sum \lambda_g g \) to the map \( x \mapsto \lambda_x \).

13.2.3. Definition. Let \( \mathbb{K} \) be a field and \( G \) be a group.

- A \( \mathbb{K} \)-linear representation of \( G \) is a \( \mathbb{K}[G] \)-module.
- A representation is irreducible if it is a simple \( \mathbb{K}[G] \)-module.
- A morphism of representations is a \( \mathbb{K}[G] \)-morphism.
- Hence two representations are isomorphic if they are, as \( \mathbb{K}[G] \)-modules.

One should pause and check that definitions do match with their naive forms of §1. The above relies on a form of universal property: every group morphism \( \rho: G \rightarrow \text{GL}(V) \) extends to a unique \( \mathbb{K} \)-algebra morphism \( \mathbb{K}[G] \rightarrow \text{End}_{\mathbb{K}}(V) \). We still denote it by \( \rho \).

13.2.4. Remark. It is possible to tensor \( \mathbb{K}[G] \)-modules over \( \mathbb{K}[G] \), but this does not give the tensor representation.

Let \( V = \text{reg} \), which is \( \mathbb{K}[G] \) as a \( \mathbb{K}[G] \)-module. Then \( V \otimes_{\mathbb{K}[G]} V = \mathbb{K}[G] = V \), while \( V \otimes_{\mathbb{K}} V \) has dimension \( (\dim V)^2 \) > \( \dim V \).

13.2.5. Theorem (Schur’s Lemma, revisited). Let \( R \) be any ring and \( V_1, V_2 \) be simple \( R \)-modules.

(i) If \( f: V_1 \rightarrow V_2 \) is an \( R \)-morphism, then either \( f = 0 \) or \( f \) is an isomorphism.

(ii) In particular, if \( V \) is a simple \( R \)-module, then \( \text{End}_R(V) \) is a skew-field.

(iii) Suppose that \( R \) is a finite-dimensional associative \( \mathbb{K} \)-algebra for some algebraically closed field \( \mathbb{K} \). If \( V \) is a finitely generated, simple \( R \)-module, then \( \text{End}_R(V) = \mathbb{K} \text{Id}_V \).

13.2.6. Definition. Let \( R \) be a ring.

- An \( R \)-module \( M \) is semisimple if it a direct sum of simple \( R \)-modules.
- \( R \) itself is said to be semisimple if it is, as a (left) \( R \)-module. See exercise 13.5.2.

13.2.7. Theorem. Let \( R \) be a semisimple ring and \( M \) be an \( R \)-module. Then:
(i) Every submodule admits a direct complement.

(ii) $M$ is a direct sum of simple submodules.

(iii) If $M$ is finitely generated, then it is a direct sum of finitely many simple submodules.

13.2.8. Theorem (Maschke’s Theorem, revisited). Let $G$ be a finite group and $\mathbb{K}$ be a field of coprime characteristic. Then $\mathbb{K}[G]$ is semisimple.

Theorem 3.3.2 on isotypical components translates so immediately that we do not reproduce it. We move to the isomorphism type of the group algebra.

13.2.9. Theorem. Let $G$ be a finite group and $\mathbb{K}$ be a good field. Let $d_\rho = \dim \rho$ be the dimensions of the irreducible representations of $G$ over $\mathbb{K}$. Then as associative $\mathbb{K}$-algebras:

$$\mathbb{K}[G] \cong \prod_{\rho \in \text{Irr}(G)} M_{d_\rho}(\mathbb{K}) \ [\mathbb{K}\text{-Alg}].$$

This is a special instance of the Artin-Wedderburn theorem.$^{13}$

**Proof.** Since $\dim V_\rho = d_\rho$, one has $V_\rho \cong \mathbb{K}^{d_\rho} \ [\mathbb{K}\text{-Mod}]$. Hence $\text{Hom}_\mathbb{K}(V_\rho, V_\rho) \cong \text{End}_\mathbb{K}(\mathbb{K}^{d_\rho}) = M_{d_\rho}(\mathbb{K}) \ [\mathbb{K}\text{-Mod}]$. For each $\rho$ we fix a basis $\mathcal{B}_\rho$ of $V_\rho$. So we have now fixed an isomorphism:

$$\prod_{\rho \in \text{Irr}(G)} \text{End}_\mathbb{K}(V_\rho) \cong \prod_{\rho \in \text{Irr}(G)} M_{d_\rho}(\mathbb{K}) \ [\mathbb{K}\text{-Alg}].$$

For $g \in G$ and $\rho \in \text{Irr}(G)$, let $M_\rho(g) = \text{Mat}_{\mathcal{B}_\rho}(g)$. Each $M_\rho(\cdot)$ is a group morphism from $G$ to $\text{GL}_{d_\rho}(\mathbb{K})$. Now to $g \in G$ associate the family of matrices:

$$(M_\rho(g))_\rho \in \prod_{\rho \in \text{Irr}(G)} M_{d_\rho}(\mathbb{K}).$$

This defines a multiplicative map $G \to \prod_{\rho \in \text{Irr}(G)} M_{d_\rho}(\mathbb{K})$, which extends by linearity to a morphism of associative $\mathbb{K}$-algebras $\mathcal{F} : \mathbb{K}[G] \to \prod_{\rho \in \text{Irr}(G)} M_{d_\rho}(\mathbb{K})$. We claim that $\mathcal{F}$ is an isomorphism in $\mathbb{K}\text{-Alg}$.

$\mathcal{F}$ is injective: if $g$ acts trivially on every $V_\rho$, then $g$ acts trivially on any representation, so it acts trivially on the regular representation. But $\text{reg}$ is injective, so $g = 1$.

Now $\mathcal{F}$ is surjective as well, because $|G| = \sum_{\rho \in \text{Irr}(G)} d_\rho^2$. It is an isomorphism.

An explicit isomorphism is discussed in §13.4. We first return to orthogonality and class functions.

13.3 Central idempotents

Let us reformulate orthonormality of the irreducible characters in abstract terms.

13.3.1. Definition. Let $R$ be a ring.

- An element $x$ is central if $\forall y (xy = yx)$.
- An idempotent is a nonzero element $e \neq 0 \in R$ with $e^2 = e$.
- Two idempotents $e_1, e_2$ are orthogonal if $e_1 e_2 = e_2 e_1 = 0$.

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$^{13}$I have to teach this in the Village someday.
• An idempotent \( e \) is primitive if it cannot be written as \( e = f_1 + f_2 \) where \( f_i \) are idempotents \( \neq 0, 1 \).

• A complete set of central primitive idempotents is a family of central, primitive idempotents summing to 1.

13.3.2. Theorem. Let \( G \) be a finite group and \( \mathbb{K} \) be a good field. For \( \chi \in \text{Irr}(\mathbb{K}[G]) \), let
\[
eq \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g.
\]
Then \( \{ e_{\chi} : \chi \in \text{Irr}(\mathbb{K}[G]) \} \) is both:

(i) a \( \mathbb{K} \)-linear basis of the centre of \( \mathbb{K}[G] \);

(ii) a complete set of central primitive idempotents of \( \mathbb{K}[G] \).

Proof. This is essentially repeating the orthogonality relations; here are the main lines.

(i) The space \( C_{\mathbb{K}}(G) \) of class functions is exactly the centre of \( \mathbb{K}[G] \). Indeed, \( f = \sum_{g} \lambda_g g \in \mathbb{K}[G] \) is central iff \( (\forall h \in G) (f h = h f) \) iff \( (\forall g, h \in G) (\lambda_{gh} = \lambda_{hg}) \).

Since characters are class functions, we do have \( e_{\chi} \in Z(\mathbb{K}[G]) \).

They form a family of cardinal \# \text{Irr} \( (\mathbb{K}[G]) = \# \text{Conj}(G) = \dim C_{\mathbb{K}}(G) = \dim Z(\mathbb{K}[G]) \). So it only remains to prove linear independence, which will follow from (ii).

(ii) We analyse \( \text{reg} \) term by term.

Let \( (W, \sigma) \) be an irreducible representation with character \( \psi \). Since \( e_{\chi} \in Z(\mathbb{K}[G]) \), one has \( \sigma(e_{\chi}) \in Z(\mathbb{K}[G]) \), so \( \sigma(e_{\chi}) \) is \( G \)-covariant. By Schur's Lemma, there is \( \lambda_{\chi, \psi} \in \mathbb{K} \) such that \( \sigma(e_{\chi}) = \lambda_{\chi, \psi} \text{Id}_W \). Now:
\[
\lambda_{\chi, \psi} \psi(1) = \lambda_{\chi, \psi} \dim W = \text{tr} \sigma(e_{\chi}) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) \psi(g) = \chi(1) (\chi | \psi).
\]

By orthogonality, \( \lambda_{\chi, \psi} = \delta_{\chi, \psi} \). Thus \( \sigma(e_{\chi}) = \lambda_{\chi, \psi} \text{Id}_W \) is \( o \) on irreducible representations non-isomorphic to \( \chi \), and \( \text{Id} \) on irreducible representations isomorphic to \( \chi \).

Since \( \text{reg} \) is a direct sum of irreducible representations, \( e_{\chi} = \text{reg}(e_{\chi}) \) is the identity on \( \text{Iso}_{\text{reg}}(\chi) \) and the zero map on other isotypical components: thus \( e_{\chi} \) is the projector onto \( \text{Iso}_{\text{reg}}(\chi) \) parallel to the other terms.

It follows that \( e_{\chi} \) is an idempotent, and \( e_{\chi} e_{\psi} = o \) whenever \( \chi \neq \psi \). Finally, \( \sum e_{\chi} = \text{Id}_{\text{reg}} = 1 \). This certainly implies linear independence, and also completes the proof of (i).

\[ \square \]

13.3.3. Remark. One sometimes uses the following \( \mathbb{Z} \)-basis of \( \mathbb{Z}[G] \). For \( \gamma \in \text{Conj}(G) \) a conjugacy class, let \( e_{\gamma} = \sum_{g \in \gamma} g \). Then \( \{ e_{\gamma} : \gamma \in \text{Conj}(G) \} \) is a \( \mathbb{Z} \)-basis of \( \mathbb{Z}[G] \). It can be proved that every \( e_{\gamma} \) is an algebraic integer of the ring \( \mathbb{K}[G] \). (This is what we did in Theorem 7.3.1, Step 1.)

(*) 13.4 Fourier transform

Let \( G \) be a finite group and \( \mathbb{K} \) be a good field. By the Artin-Wedderburn theorem (Theorem 13.2.9), there are finite-dimensional vector spaces \( V_1, \ldots, V_r \) and an isomorphism
of associative $\mathbb{K}$-algebras:

$$\mathbb{K}[G] \simeq \prod_{i=1}^{r} \text{End}_{\mathbb{K}}(V_i) \quad [\mathbb{K} \cdot \text{Alg}].$$

(We do mean $\text{End}_{\mathbb{K}}$ and certainly not $\text{End}_{\mathbb{K}[G]}$.) The present, completely optional, subsection elaborates on this fact and gives an explicit isomorphism: the Fourier transform on the group.

**Notation.** Even though we work over an arbitrary good field, we adopt the analytic point of view on the group algebra. So $\mathbb{K}[G]$ is the $\mathbb{K}$-algebra of functions $f: G \to \mathbb{K}$, equipped with the convolution product

$$(f_1 \ast f_2)(g) = \sum_{x, y \in G} f_1(x) f_2(y).$$

(In algebraic terms, if $f = \sum \lambda_g g$, one simply lets $f(g) = \lambda_g$.)

Thus $f(g)$ makes sense, and is a scalar. Below we shall also consider functions $\varphi: \text{Irr}_{\mathbb{K}}(G) \to S$ for some set $S$. Then $\varphi(\rho)$ will make sense. (Here, $\varphi$ will not stand for a linear form.)

The Fourier transform takes functions $f$ (viz. functions from $G$ to $\mathbb{K}$) to functions $\varphi$ (viz. certain functions from $\text{Irr}_{\mathbb{K}}(G)$ to $S$). The inverse Fourier transform does the converse.

**The general case.** In physics the Fourier coefficients are not pure numbers. At each frequency one has an energy; these quantities have very different natures; energies are parametrised by frequencies. In non-abelian group theory, the Fourier 'coefficients' are:

- linear operators,
- parametrised by irreducible representations.

Hence $\text{Irr}_{\mathbb{K}}(G)$ plays the role of the spectrum, viz. the set indexing the Fourier components, and each component is in $\text{End}_{\mathbb{K}}(\rho)$.

13.4.1. **Notation.**

- For $\rho: G \to \text{GL}(V)$ a representation of $G$ we still denote by $\rho$ its natural extension $\mathbb{K}[G] \to \text{End}_{\mathbb{K}}(V)$, which is a morphism of $\mathbb{K}[G]$-algebras.

Hence, always seeing $f$ as a function $G \to \mathbb{K}$:

$$\rho(f) = \sum_{g \in G} f(g) \rho(g) \quad \in \text{End}_{\mathbb{K}}(V).$$

Notice that $\rho(f) \in \text{End}_{\mathbb{K}}(V)$; in general $\rho(f)$ need not be $G$-covariant.

- For $f \in \mathbb{K}[G]$ we let: $\hat{f}(\rho) = \rho(f) = \sum_{g \in G} f(g) \rho(g)$.

13.4.2. **Remark.** Technically, $\hat{f}$ is thus a map from $\text{Irr}_{\mathbb{K}}(G)$ to $\bigsqcup_{\rho \in \text{Irr}_{\mathbb{K}}(G)} \text{End}_{\mathbb{K}}(V_{\rho})$.

However, at each $\rho$, one has $\hat{f}(\rho) \in \text{End}_{\mathbb{K}}(V_{\rho})$, and therefore:

$$\hat{f} \in \prod_{\rho \in \text{Irr}_{\mathbb{K}}(G)} \text{End}_{\mathbb{K}}(V_{\rho}).$$

(If $G$ is abelian, this will simplify dramatically.)
Theorem. Let $G$ be a finite group and $\mathbb{K}$ be a good field.

(i) For $f \in \mathbb{K}[G]$ let $\hat{f}(\rho) = \sum_{g \in G} f(g) \rho(g)$. Then $\mathcal{F}(f) = \hat{f}$ defines an isomorphism of $\mathbb{K}[G]$-algebras:

$$\mathbb{K}[G] \cong \prod_{\rho \in \text{Irr}(G)} \text{End}_{\mathbb{K}}(\rho) \quad [\text{K-Alg}].$$

(ii) For $\varphi \in \prod_{\rho \in \text{Irr}(G)} \text{End}_{\mathbb{K}}(\rho)$, let $\hat{\varphi} = \mathcal{F}^{-1}(\varphi)$ be the inverse. Then:

$$\hat{\varphi}(g) = \sum_{\rho \in \text{Irr}(G)} \dim \rho \left( \frac{1}{|G|} \text{tr} \left[ \rho(g^{-1}) \varphi(\rho) \right] \right).$$

Remarks.

• We always consider $\text{End}_{\mathbb{K}}(\rho)$, since $\text{End}_{\mathbb{K}[G]}(\rho)$ is only 1-dimensional by Schur’s Lemma. (Cf. abelian case below.)

• $\mathcal{F}$ is a morphism of associative $\mathbb{K}$-algebras, but not a morphism of $\mathbb{K}[G]$-modules.

Proof. For brevity we let $\Pi = \prod_{\rho \in \text{Irr}(G)} \text{End}_{\mathbb{K}}(\rho)$.

(i) Clearly $\mathcal{F}: \mathbb{K}[G] \to \Pi$ is well-defined, and linear. Recall that $\mathbb{K}[G]$ is equipped with the convolution product (which extends the group law on $G$), and $\Pi$ is equipped with its Cartesian product structure, viz. we compose componentwise: $(\varphi_1 \cdot \varphi_2)(\rho) = \varphi_1(\rho) \circ \varphi_2(\rho)$. So $\mathcal{F}$ is multiplicative and takes 1 to $(1_{\rho} : \rho \in \text{Irr}(G))$. Hence it is a morphism of $\mathbb{K}$-algebras.

Now dimensions match, so it suffices to prove injectivity. Suppose $\mathcal{F}(f) = 0$. Then in each irreducible representation, (the image of) $f$ acts trivially. But the regular representation is a direct sum of such, so $f$ acts trivially on $\text{reg} \cong \mathbb{K}[G]$. Since $\mathbb{K}[G]$ has a unit (viz. since $\text{reg}$ is injective), this means $f = 0$.

(ii) This may look difficult but the formula is given, so it is a simple matter of checking it. The linear map $\mathcal{F}^{-1}$ is well-defined. Let:

$$\mathcal{G}(\varphi) = \left[ g \mapsto \sum_{\rho \in \text{Irr}(G)} \dim \rho \left( \frac{1}{|G|} \text{tr} \left[ \rho(g^{-1}) \varphi(\rho) \right] \right) \right],$$

which is clearly linear from $\Pi$ to $\mathbb{K}[G]$. We want to prove $\mathcal{G} = \mathcal{F}^{-1}$.

Let $h \in G$. When seen in $\mathbb{K}[G]$, viz. when seen as a function $G \to \mathbb{K}$, $h$ is the 'Dirac mass' $\delta_h(x) = \delta_{h,x}$, which equals 1 at $h$ and 0 everywhere else.
Then $\mathcal{F}(\delta_h) = \{ \rho \mapsto \rho(h) \}$, so at any $g \in G$ one has:

$$G(\mathcal{F}(\delta_h))(g) = \sum_{\rho \in \text{Irr}_K(G)} \frac{\dim \rho}{|G|} \cdot \text{tr} (\rho(g^{-1})\rho(h)) = \sum_{\rho \in \text{Irr}_K(G)} \frac{1}{|G|} \cdot \dim \rho \cdot \chi_\rho(g^{-1}h) = \frac{1}{|G|} \text{reg}_G(g^{-1}h) = \frac{|G|}{|G|} \delta_{g^{-1}h}(1) = \delta_h(g).$$

This also equals $\mathcal{F}^{-1}(G(\mathcal{F}(\delta_h)))(g)$, and equality holds at every $g$. Thus, as functions, $G(\mathcal{F}(\delta_h)) = \mathcal{F}^{-1}(G(\mathcal{F}(\delta_h)))$. Therefore $G$ and $\mathcal{F}^{-1}$ agree on $\{ \mathcal{F}(\delta_h) : h \in G \}$, which is a basis of $\Pi$. By linearity, they agree everywhere. 

**The abelian case.** Let $A$ be a finite abelian group. 

If $\rho \in \text{Irr}_K(A)$ then $\dim \rho = 1$. Although a 1-dimensional vector space $V$ is non-canonically isomorphic to $K$, the ring $\text{End}(V)$ is canonically isomorphic to $K$. Here $\rho: A \to \text{GL}(V)$, so we may assume $\rho: A \to K^\times$. Thus irreducible representations are elements of the dual group $\hat{A}$, viz. the group of all morphisms $A \to K^\times$. Hence, here, $\text{Irr}_K(A) = \hat{A}$ bears an additional group structure.

Return to $\rho \in \hat{A}$ and to the canonical isomorphism $\text{End}_K(\rho) \simeq K$. Therefore:

$$\prod_{\rho \in \text{Irr}_K(A)} \text{End}_K(\rho) \simeq \prod_{A} K \simeq [K^{\hat{A}} K^{\text{Alg}}].$$

The right-hand is no longer formed of operators but of Fourier coefficients. 

13.4.5. **Remark.** Be careful that $K^{\hat{A}}$, as a $K$-algebra, is still equipped with the componentwise product, which is not the same as the convolution product. (This simply means $K^\hat{A} \neq [K^\hat{A}] K^{\text{Alg}}$.) There is no escaping from the fact that Fourier changes convolution to componentwise, and vice-versa.

The inverse transform rewrites as follows. Since every linear endomorphism $u$ of a 1-dimensional vector space is the scalar action of $\lambda_u = \text{tr} u$, the inverse transform formula simplifies to:

$$\hat{\phi}(a) = \sum_{\rho \in \hat{A}} \frac{1}{|A|} \rho(-a) \phi(\rho).$$

(**) 13.4.6. **Remark.** First notice that the direct transform can rewrite as:

$$\hat{f}(\rho) = \sum_{a \in A} f(a) \rho(a) = |A| \langle f \rho \rangle_A,$$

where $\langle \cdot | \cdot \rangle_A$ is the usual bilinear form on $C_0(A)$.

With perversity one may move to the bidual group $\hat{\hat{A}}$ and rewrite the inverse transform using it. Let $\text{ev}_u: \hat{\hat{A}} \to K^\times$ take $\rho$ to $\text{ev}_u(\rho) = \rho(a)$ (‘evaluation map’). It is not hard to see that $\text{ev}: a \mapsto \text{ev}_a$ is a natural isomorphism $A = \hat{\hat{A}}$. The $\text{ev}_a$ are the irreducible characters of $\hat{\hat{A}}$. 

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Then the inverse formula de-simplifies to:

\[ \psi(a) = \sum_{\rho \in \hat{A}} \frac{1}{|\hat{A}|} (\rho(-a)) \psi(\rho) = \sum_{\rho \in \hat{A}} \frac{1}{|\hat{A}|} (\text{ev}_a(-\rho)) \psi(\rho) = (\text{ev}_a | \psi)_{\hat{A}}, \]

where \((\cdot | \cdot)_{\hat{A}}\) is the usual bilinear form on \(C_\mathbb{K}(\hat{A})\).

### 13.5 Exercises

**13.5.1. Exercise.** Let \( \mathbb{K} \) be a field, \( G_1, G_2 \) be two groups, and \( \varphi: G_1 \to G_2 \) be a group morphism.

(i) Show that \( \varphi \) naturally induces a morphism of \( \mathbb{K} \)-algebras \( \Phi: \mathbb{K}[G_1] \to \mathbb{K}[G_2] \).

(ii) Show that \( \varphi \) is an isomorphism iff \( \Phi \) is an isomorphism.

**Note.** It is an open problem in group theory whether one can have \( \mathbb{K}[G_1] \cong \mathbb{K}[G_2] \) as abstract \( \mathbb{K} \)-algebras without having \( G_1 \cong G_2 \) as groups.

**13.5.2. Exercise.** Let \( R \) be a ring. Prove that \( R \) is semisimple as a left \( R \)-module iff it is as a right \( R \)-module.

**13.5.3. Exercise.** Let \( A \) be a finite abelian group.

1. Let \( B \leq A \) be a subgroup. Prove that every morphism \( B \to \mathbb{C}^* \) extends to \( A \).

2. Prove that \( \hat{A} \) is isomorphic to \( A \), and that \( \hat{\hat{A}} \) is canonically isomorphic to \( A \).

**Further reading**


