Variation formulae for submanifolds in Kähler and G_2 geometry

Tommaso Pacini*

*Dept. of Mathematics, University of Torino, Italy
tommaso.pacini@unito.it

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1 Introduction

The goal of these notes is to (i) present the first and second variation formulae for submanifolds in various geometric contexts, (ii) comment on the necessary formalism, (iii) provide some indication of the larger picture related to some of this geometry.

Section 2 examines the simplest case: curves. This case serves as a good introduction to the issues at stake. Section 3 provides the general formulae in Riemannian manifolds. Section 4 provides a digression on another important way to ensure the stability of a submanifold. Sections 5, 6 explain how the second variation formula improves for certain types of submanifolds in the Kähler and G_2 context.

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2 Variational formulae for curves

Toolbox. Let M be a smooth manifold. Recall that there is no problem in differentiating functions $f : M \to \mathbb{R}$. Given $X \in T_pM$, the directional derivative of f can be obtained, for example, via any curve α tangent to X:

$$Xf = df(X) = \frac{d}{dt}(f \circ \alpha)(t)_{|t=0}.$$

It is important to notice that second derivatives generally do not commute. Let X, Y be vector fields. Then, locally, $X(Yf)$ contains derivatives of the coefficients of Y, while $Y(Xf)$ contains derivatives of the coefficients of X. The difference is encoded by the Lie bracket $[X,Y] := XY - YX$. This is again a vector field. It vanishes iff there exist local coordinates such that X, Y can be written using constant coefficients. More geometrically: iff X, Y can be identified with the vector fields $\partial x, \partial y$ defined by a local immersion $\mathbb{R}^2 \to M$.

A smooth manifold does not provide enough structure to differentiate vector fields Z. For this we need a connection. Any metric g on M provides a special connection, called the Levi-Civita connection. The derivative of Z in the direction $X \in T_pM$ is written $\nabla_X Z$. The Levi-Civita connection has two important properties:

- Compatibility with g: $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.
- Compatibility with first-order calculus: $\nabla_X Y - \nabla_Y X = [X,Y]$.

Once again, given two vector fields X, Y, second derivatives generally do not commute, even after discounting the issue of variable coefficients. The additional problem is due to the derivatives of the coefficients of the metric. The difference is encoded by the Riemann curvature tensor

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature tensor has several symmetries. In particular:

- $R(X,Y)Z = -R(Y,X)Z$,
- $R(X,Y)Z \cdot W = -R(X,Y)W \cdot Z$,
- $R(X,Y)Z \cdot W = R(Z,W)X \cdot Y$.

It follows that $R'(X \wedge Y, W \wedge Z) := R(X,Y)Z \cdot W$ extends to a symmetric bilinear form

$$R' : \Lambda^2(TM) \times \Lambda^2(TM) \to \mathbb{R}.$$

We say that R is positive if R' is positive in the sense of bilinear forms: $R'(X \wedge Y, X \wedge Y) = R(X,Y)Y \cdot X \geq 0$ for each X, Y. The notion of negativity is analogous. Both conditions are very strong: the typical situation is for R to be positive in certain directions, negative in others.

This definition of curvature is very analytic. A large part of Riemannian geometry is devoted to understanding the geometric consequences of properties of R. We shall be exploring how certain positivity/negativity properties of R influence the properties of the length/volume functionals for curves/submanifolds.

Remark. Beginning courses concerning surfaces Σ in $\mathbb{R}^3$ define curvature using the Gauss map. The Gauss map is however actually related to the second fundamental form. The equivalence between the two definitions of curvature is purely accidental: it is a consequence of the Gauss equation, in the form:

$$\text{curv of } M = \text{curv of } \Sigma + \text{second fund form}.$$

Since $M = \mathbb{R}^3$ is flat, the LHS vanishes: curv of Σ=-(second fund form).

The definition of R' ensures that the two notions of curvature lead to the same notion of positivity.

From the viewpoint of Physics, the fundamental role of a connection is to define the notion of acceleration $\nabla_{\dot\alpha}\dot\alpha$ of a curve $\alpha(t)$, thus providing the necessary language to encode Newton's equation $F = ma$. A curve is called a geodesic if $\nabla_{\dot\alpha}\dot\alpha = 0$: such curves correspond to inertial motions within M. We stress the fact that, both by definition and because of their desired physical interpretation, geodesics are curves with a parametrization.

First variation formula. Let $\alpha(t)$ be a curve in (M, g). Recall that its length, over an interval $[a, b]$, is defined by $L(\alpha) := \int_a^b |\dot\alpha| dt$. It is independent of the specific parametrization of α. According to the context one is usually interested in either the length functional for closed curves, or the length functional for curves with fixed end-points.

Let us consider, for example, the length functional L on the space of closed curves $\alpha : \mathbb{S}^1 \to M$. A variation of α is a family of such curves $\alpha(t, \tau)$. The corresponding infinitesimal variation is the vector field $Z(t) := \frac{\partial\alpha}{\partial\tau}(t, 0)$.

Remark. The manifold $\mathbb{S}^1$ admits global coordinates of the form $t \in [a, b]/\sim$. Compared to the case of submanifolds, this simplifies the formalism. Of course it also admits the canonical coordinate $\theta \in [0, 2\pi]/\sim$, but we will not need this.

Notation. We shall write $\dot\alpha = \alpha_t = \frac{\partial\alpha}{\partial t}$, $\alpha_\tau = \frac{\partial\alpha}{\partial\tau}$. When applying the metric to vectors, we shall write $(v, w) = v \cdot w = g(v, w)$.

Proposition 2.1. *Let α be a closed curve in (M, g). Let Z be an infinitesimal variation and e be a global unit vector field tangent to α. Then*

$$dL_{|\alpha}(Z) = \frac{d}{d\tau}(L \circ \alpha)(\tau)_{|\tau=0} = -\int_a^b (Z, \nabla_e e)|\dot\alpha| dt.$$

To prove this, we shall start with the following observation:

$$\begin{aligned}
\frac{d}{d\tau}\sqrt{|\alpha_t|^2}_{|\tau=0} &= (1/2)\frac{1}{|\alpha_t|}\frac{\partial}{\partial\tau}(\alpha_t, \alpha_t)_{|\tau=0} \\
&= \frac{1}{|\alpha_t|}(\nabla_{\alpha_\tau}\alpha_t, \alpha_t) = (\nabla_{\alpha_t}\alpha_\tau, \alpha_t/|\alpha_t|) \\
&= \frac{\partial}{\partial t}(\alpha_\tau, \alpha_t/|\alpha_t|) - (\alpha_\tau, \nabla_{\alpha_t}(\alpha_t/|\alpha_t|)) \\
&= \frac{\partial}{\partial t}(Z, e) - (Z, \nabla_e e)|\alpha_t|,
\end{aligned}$$

where we use the fact that $\nabla_{\alpha_\tau}\alpha_t - \nabla_{\alpha_t}\alpha_\tau = [\alpha_\tau, \alpha_t] = 0$. Thus

$$\begin{aligned}\frac{d}{d\tau}(L\circ\alpha)(\tau)_{|\tau=0} &= \frac{d}{d\tau}(\int_a^b \sqrt{|\alpha_t|^2}dt)_{|\tau=0} = \int_a^b \frac{d}{d\tau}\sqrt{|\alpha_t|^2}_{|\tau=0}dt\\ &= -\int_a^b (Z, \nabla_e e)|\dot{\alpha}|dt.\end{aligned}$$

Notice that, in this formula, each element Z, $\nabla_e e$, $|\dot{\alpha}|dt$ is geometrically well-defined, indipendent of the parametrization. This should be expected: the functional is invariant under reparametrization, so its critical points should also be parametrization-independent. Critical points can thus be viewed as certain non-parametrized curves.

From this viewpoint, these critical points are very different from geodesics, which by definition are parametrized. Notice that the intuitive meaning of geodesics and of critical points of L is also rather different. That said, using appropriate care one can still find a relationship between geodesics and critical points of the length functional.

Corollary 2.2. *The curve α is a critical point for L iff $\nabla_e e \equiv 0$.*

In particular, each (closed) geodesic is a critical point for L. Conversely, if α is a critical point for L and we parametrize it so that $|\dot{\alpha}|$ is constant, then α is a geodesic.

To prove the first statement, notice that the equation tells us that α is a critical point iff the normal component $H := \nabla_e^\perp e$ vanishes. However, since $|e| \equiv 1$, $\nabla_e e$ is necessarily orthogonal to the curve: $\nabla_e^\perp e = \nabla_e e$. To prove the remaining statements we can use the fact that if $\alpha(t)$ is a geodesic then $|\dot{\alpha}|$ is constant, so $\dot{\alpha}$ is a constant multiple of e.

Remark. One can show that geodesics are precisely the critical points of a different functional, energy, which depends on the parametrization.

Second variation formula. Assume α is a critical point for L. We would like to know whether it is a local minimum. We need to calculate the second derivatives of $L\circ\alpha$ wrt all variations of α.

The starting point is the fact that, with the obvious extensions of the definitions of Z, e, the first variation formula

$$\frac{d}{d\tau}(L\circ\alpha)(\tau) = -\int_a^b (Z, \nabla_e e)|\dot{\alpha}|dt$$

holds for all τ. We want to differentiate it again. Considering that L is parametrization-invariant, thus invariant wrt tangential variations, we can restrict our attention to normal variations, ie assume that $Z\cdot\dot{\alpha} \equiv 0$.

$$
\begin{aligned}
\frac{d^2}{d\tau^2}(L\circ\alpha)(\tau)_{|\tau=0} &= -\frac{d}{d\tau}\int_a^b (Z,\nabla_e e)|\dot\alpha|dt_{|\tau=0}\\
&= -\int_a^b (\nabla_Z Z,\nabla_e e)|\dot\alpha|dt - \int_a^b (Z,\nabla_Z\nabla_e e)|\dot\alpha|dt\\
&\quad + \int_a^b (Z,\nabla_e e)^2|\dot\alpha|dt\\
&= -\int_a^b (Z,\nabla_Z\nabla_e e)|\dot\alpha|dt,
\end{aligned}
$$

where we use a previous calculation for $d/d\tau(|\dot\alpha|)$ and the fact that, since α is a critical point, $\nabla_e e = 0$. Notice also that

$$
\begin{aligned}
[Z,e] &= [Z,\dot\alpha/|\dot\alpha|] = Z(1/|\dot\alpha|)\dot\alpha + (1/|\dot\alpha|)[Z,\dot\alpha]\\
&= -(Z(|\dot\alpha|)/|\dot\alpha|)e,
\end{aligned}
$$

because $[Z,\dot\alpha] = [\alpha_\tau,\alpha_t] = 0$. It follows that

$$
\begin{aligned}
-\int_a^b (Z,\nabla_Z\nabla_e e)|\dot\alpha|dt &= -\int_a^b (Z,\nabla_e\nabla_Z e)|\dot\alpha|dt - \int_a^b (Z,\nabla_{[Z,e]}e)|\dot\alpha|dt\\
&\quad - \int_a^b (Z,R(Z,e)e)|\dot\alpha|dt\\
&= -\int_a^b (Z,\nabla_{\dot\alpha}\nabla_Z e)dt + \int_a^b (Z(|\dot\alpha|)/|\dot\alpha|)(Z,\nabla_e e)|\dot\alpha|dt\\
&\quad - \int_a^b (Z,R(Z,e)e)|\dot\alpha|dt\\
&= -\int_a^b (Z,\nabla_{\dot\alpha}\nabla_Z e)dt - \int_a^b (Z,R(Z,e)e)|\dot\alpha|dt\\
&= -\int_a^b \frac{d}{dt}(Z,\nabla_Z e)dt + \int_a^b (\nabla_{\dot\alpha}Z,\nabla_Z e)dt\\
&\quad - \int_a^b (Z,R(Z,e)e)|\dot\alpha|dt\\
&= \int_a^b (\nabla_e Z,\nabla_Z e)|\dot\alpha|dt - \int_a^b (Z,R(Z,e)e)|\dot\alpha|dt.
\end{aligned}
$$

Also,

$$
\begin{aligned}
(\nabla_e Z,\nabla_Z e) &= (\nabla_e Z,\nabla_e Z + [Z,e])\\
&= (\nabla_e Z,\nabla_e Z) - (Z(|\dot\alpha|)/|\dot\alpha|)(\nabla_e Z,e)\\
&= (\nabla_e Z,\nabla_e Z) + (Z(|\dot\alpha|)/|\dot\alpha|)(Z,\nabla_e e)\\
&= (\nabla_e Z,\nabla_e Z).
\end{aligned}
$$

We have thus proved the following.

Theorem 2.3. *Let α be a critical point for L. Then, wrt any normal variation,*

$$\frac{d^2}{d\tau^2}(L\circ\alpha)(\tau)_{|\tau=0} = \int_a^b (\nabla_e Z, \nabla_e Z)|\dot\alpha|dt - \int_a^b (R(Z,e)e, Z)|\dot\alpha|dt.$$

This formula shows us that curvature can be used to force critical points to have interesting behaviour.

Corollary 2.4. *Assume (M,g) has non-positive curvature. Then any critical point for L is a local minimum.*

Example. Any great circle (suitably parametrized) on a sphere is a geodesic. The fact that curvature is positive corresponds to the fact that the curve can be shrunk, eg via parallels, to curves of shorter length.

As mentioned, similar results hold for curves between two fixed points. Choosing two points p, q on a great circle on a sphere, close to each other, it is again clear that the longer of the two geodesic arcs between them is not a local minimum. Consider instead two points on a (flat) cylinder. In this case, as predicted by the theorem, any geodesic between them is a local minimum, but not necessarily a global minimum.

Remark. The second variation formula may alternatively be written as

$$\frac{d^2}{d\tau^2}(L\circ\alpha)(\tau)_{|\tau=0} = -\int_a^b (\nabla_e\nabla_e Z + R(Z,e)e, Z)|\dot\alpha|dt,$$

using the fact that $\frac{d}{dt}(\nabla_e Z, Z) = (\nabla_{\dot\alpha}\nabla_e Z, Z) + (\nabla_e Z, \nabla_{\dot\alpha} Z)$.

The bigger picture. The second variation formula is very useful for studying the topology of certain manifolds. Assume (M,g) is complete and has strictly positive curvature, in the sense that $R \geq \delta > 0$. Given any two points $p, q \in M$ and a geodesic between them, Bonnet and Synge showed how to construct a special infinitesimal deformation to prove that the distance $d(p,q)$ is bounded above by a constant. In other words, any such M has bounded diameter. The Hopf-Rinow theorem then shows that M is compact. Applying this argument to the universal cover $\tilde M$ of M shows that $\tilde M$ is compact, so M has finite fundamental group. Myers showed how to improve this argument to obtain the same result using only the assumption $\mathrm{Ric} \geq \delta > 0$.

If M is semi-Riemannian, length and distance are no longer well-defined so it is more useful to study the energy functional $\int |\dot\alpha|^2 dt$. As mentioned, this functional depends on the parametrization of α.

3 Variation formulae for submanifolds

Toolbox. Let Σ be an abstract k-dimensional manifold, endowed with an immersion $\phi : \Sigma \to (M,g)$. We shall often identify Σ with its image. The metric on Σ defined by the restriction of g induces a Levi-Civita connection ∇^Σ for Σ. One can check that it coincides with the tangential projection of the Levi-Civita connection ∇^M of M: given any vector fields X, Y tangent to Σ,

$$\nabla^\Sigma_X Y = (\nabla^M_X Y)^T.$$

The perpendicular component $(\nabla^M_X Y)^\perp$ is known as the second fundamental form. It is a tensor of the form $T\Sigma \times T\Sigma \to (T\Sigma)^\perp$. A submanifold is called totally geodesic if the second fundamental form vanishes. This is a very strong condition. It implies, for example, that geodesics in Σ are also geodesics in M.

Example. The image of any geodesic in M is a 1-dimensional totally geodesic submanifold.

The trace of the second fundamental form

$$H := \mathrm{tr}_g(\nabla^\perp) = g^{ij}(\nabla^M_{X_i} X_j)^\perp = (\nabla^M_{e_i} e_i)^\perp$$

is known as the mean curvature vector field: it is a normal vector field along Σ.

We shall often use the same notation ∇ for both ∇^Σ and ∇^M.

Example. Consider the first variation formula for a curve α. The fact $(e, e) \equiv 1$ implies that $(e, \nabla_e e) \equiv 0$, so we may write

$$\frac{d}{d\tau}(L \circ \alpha)_{|\tau=0} = -\int_a^b (Z^\perp, (\nabla_e e)^\perp)|\dot\alpha| dt = -\int_a^b (Z^\perp, H)|\dot\alpha| dt.$$

This agrees with the fact that L is parametrization-invariant, so it does not detect the tangential deformation corresponding to Z^T.

First variation formula. Let Σ be as above. Local coordinates $\psi : U \subseteq \mathbb{R}^k \to \Sigma$ provide a local volume form on the image; pulled back to U, it takes the form $\sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^k$, where $g_{ij} = g(d\psi(\partial x_i), d\psi(\partial x_j))$. If Σ is oriented and we choose a compatible atlas, these local volume forms glue to define a global volume form which we shall denote vol_g. The volume of Σ is then $\int_\Sigma \mathrm{vol}_g$.

Assume Σ is compact, so that the volume is finite. Let us consider the volume functional Vol on the space of immersions $\phi : \Sigma \to M$. A variation of an immersion is a family $\phi(\tau)$. The corresponding infinitesimal variation is the vector field $Z := \frac{\partial \phi}{\partial \tau}$. Calculations similar to those above lead to the following first variation formula.

Proposition 3.1. *Let $\phi : \Sigma \to M$ be a compact oriented immersed submanifold. Let Z be an infinitesimal variation. Then*

$$d\,\mathrm{Vol}_{|\phi}(Z) = \frac{d}{d\tau}(\mathrm{Vol} \circ \phi)(\tau)_{|\tau=0} = -\int_\Sigma (Z, H)\,\mathrm{vol}_g.$$

It follows that Σ (more precisely: ϕ) is a critical point for Vol *iff $H = 0$.*

A submanifold is called minimal if it is critical, ie $H = 0$. This terminology is standard but can be misleading: we must distinguish minimal submanifolds from those which are local/global minima of the volume functional: these submanifolds are also called stable critical points.

Remark. As in the case of curves, the condition $H = 0$ is parametrization-invariant, ie it depends only on the image submanifold: this is coherent with the fact that Vol is parametrization-invariant. The notion of minimality can be related to harmonic maps, ie critical points of the energy functional. This notion requires a fixed metric on Σ. One finds that (the image of) every harmonic map is minimal; conversely, every minimal submanifold, appropriately parametrized (via a conformal immersion) is a harmonic map.

Second variation formula. As before, we are interested in finding conditions ensuring that a minimal submanifold is a local minimum. Once again, we need to differentiate the first variation formula. Careful manipulation of the various terms leads to the following result.

Theorem 3.2. *Assume Z normal and Σ (more precisely: ϕ) minimal. Then*

$$\frac{d^2}{dt^2}(\mathrm{Vol}\circ\phi_\tau)_{|t=0} = \int_\Sigma (-(\nabla_{e_i}Z\cdot e_j)^2 - R(e_i,Z)Z\cdot e_i + (\nabla_{e_i}Z\cdot f_j)^2)\ \mathrm{vol}_g,$$

where $e_1,\dots,e_k$ is a orthonormal basis of $T_p\Sigma$ at any given point, $f_1,\dots,f_{n-k}$ is a orthonormal basis of $T_p\Sigma^\perp$ and R is the curvature tensor of M.

Basically, the first term is the norm squared of $(\nabla Z)^T$ (restricted to Σ), the second term is the trace along Σ of the appropriate curvature tensor and the third term is the norm squared of $(\nabla Z)^\perp$ (restricted to Σ). This explains why the expression is independent of the chosen bases.

Corollary 3.3. *Assume (M,g) has non-positive curvature. Then any totally geodesic Σ is a local minimum for* Vol.

Remark. The second variation formula may be alternatively written

$$\frac{d^2}{dt^2}(\mathrm{Vol}\circ\phi_\tau)_{|t=0} = -\int_\Sigma ((\nabla_{e_i}Z\cdot e_j)^2 + R(e_i,Z)Z\cdot e_i + (\iota_{e_i}\iota_{e_i}\nabla^\perp\nabla^\perp Z, Z))\ \mathrm{vol}_g,$$

where $\nabla^\perp$ is the natural connection on $T\Sigma^\perp$. This uses the fact that, in normal coordinates,

$$\begin{aligned}(\nabla_{e_i}Z\cdot f_j)^2 &= (\nabla^\perp_{e_i}Z, \nabla^\perp_{e_i}Z)\\ &= \nabla_{e_i}(\nabla^\perp_{e_i}Z, Z) - (\nabla^\perp_{e_i}\nabla^\perp_{e_i}Z, Z)\\ &= (1/2)\Delta(\|Z\|^2) - (\iota_{e_i}\iota_{e_i}\nabla^\perp\nabla^\perp Z, Z).\end{aligned}$$

4 Digression: calibrated geometry

The above results show how to use curvature conditions to ensure that certain submanifolds are local minima for the volume functional. It is of course particularly interesting to detect global minima, but this cannot be done only using variational formulae. Luckily there exists a completely different theory, known as calibrated geometry, which can ensure this condition.

Let (M, g) be a Riemannian manifold. A k-dimensional calibration is a differential form $\alpha \in \Lambda^k(M)$ such that, for any ON vectors $\{e_1, \ldots, e_k\} \in T_pM$, $|\alpha(e_1, \ldots, e_k)| \leq 1$. A k-dimensional submanifold Σ is calibrated if any ON basis of any $T_p\Sigma$ achieves the equality: $|\alpha(e_1, \ldots, e_k)| = 1$. Such submanifolds are automatically orientable: choosing bases such that $\alpha(e_1, \ldots, e_k) = 1$ provides an orientation. In this case $\alpha_{|T\Sigma} = \mathrm{vol}_g$. In other words, α provides a global extension of vol_g, from Σ to M.

Proposition 4.1. *Assume α is a closed calibration: $d\alpha = 0$. Then any compact calibrated submanifold is a global minimizer for* Vol *within its homology class.*

Indeed, let Σ' be any other submanifold within the same homology class. Then

$$\mathrm{Vol}(\Sigma) = \int_\Sigma \alpha = \int_{\Sigma'} \alpha \leq \int_{\Sigma'} \mathrm{vol}_g = \mathrm{Vol}(\Sigma').$$

Although very simple, this is an extremely strong result: minimizing Vol within homology is much stronger than minimizing wrt variations, ie within a homotopy class.

Notice that, if M is compact, any k-dimensional form can be normalized to ensure the above condition. However, in general it will not admit calibrated submanifolds. Furthermore, the normalization might destroy the condition $d\alpha = 0$. In this sense it is non-trivial to find useful calibrations. Their existence often depends on special geometric features of (M, g).

We shall be interested in the following two examples.

Example. Let (M, J, ω) be a Kähler manifold. This means:

- $J : TM \to TM$ satisfies $J^2 = -Id$ so that, at each point, (T_pM, J) is modelled on $(\mathbb{C}^n, i)$ and $\dim(M) = 2n$.
- $\omega \in \Lambda^2(M)$, at each point, can be identified with the standard symplectic form on $\mathbb{R}^{2n}$
$$e^{12} + e^{34} + \cdots + e^{(2n-1)2n}.$$
Here, we use the notation $e^{12} = dx^1 \wedge dy^1$, etc.
- Algebraic compatibility conditions: we ask that J preserve ω and that $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be a metric.
- Analytic integrability conditions: we ask that J correspond to local holomorphic coordinates and that $d\omega = 0$.

Then ω is a closed calibration. Its calibrated submanifolds are the complex curves. More generally, $\omega^k \in \Lambda^{2k}(M)$ is a closed calibration. Its calibrated submanifolds are the complex k-dimensional submanifolds.

Example. Let (M, φ) be a G_2 manifold. This means:

- M has dimension 7 and $\varphi \in \Lambda^3(M)$, at each point, can be identified with the form on $\mathbb{R}^7$

  $$\varphi_{std} := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{356} - e^{347}.$$

  Linear algebra then implies that each T_pM has a metric g and an orientation. This form is related to the multiplication of octonions $\mathbb{O}$: $x \times y := \mathrm{Im}(xy)$ defines a vector product on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$, and $\varphi_{std}(x, y, z) = g(x \times y, z)$.

- Analytic integrability condition: we ask that $d\varphi = 0$.

Then φ is a closed calibration. Its calibrated submanifolds are 3-dimensional submanifolds, known as associative.

Calibrated geometry thus provides complete information for certain types of submanifolds, within certain geometric contexts. Our goal in the next sections is to go back to variational methods and show that these same ambient geometries, applied to other types of submanifolds, allow us to also improve our control over variational formulae. In this way we can learn something new about these other types of submanifolds.

5 Lagrangian submanifolds

Toolbox. We shall need additional facts concerning both Riemannian and Kähler geometry.

1. Let (M, g) be a Riemannian manifold. The Ricci tensor is the tensor of type $T^*M \otimes T^*M$ defined by $\mathrm{Ric}(X, Y) := \mathrm{tr}_g R'(X, \cdot, \cdot, Y) = R(X, e_i)Y \cdot e_i$. The symmetries of R imply that Ric is a symmetric tensor, so we can define its positivity in the usual way: $\mathrm{Ric}(X, X) \geq 0$. This is a weaker notion of positivity than that of R.

The Ricci tensor appears in several analytic and geometric contexts. We are interested in its role in relating two different notions of Laplacian on differential forms. Let M be oriented. The Hodge Laplacian is $\Delta\alpha := (dd^* + d^*d)\alpha$. The covariant Laplacian is $\mathrm{tr}_g(\nabla\nabla\alpha) = \iota_{e_i}\iota_{e_i}(\nabla\nabla\alpha)$. We shall only need the case of 1-forms $\xi = g(X, \cdot)$. The Weitzenböck identity shows that

$$\Delta\xi = -\,\mathrm{tr}_g(\nabla\nabla\xi) + \mathrm{Ric}(X, \cdot).$$

2. Let (M, g) be a Kähler manifold. Then $\nabla J = 0$, so $R(X, Y)JZ = J(R(X, Y)Z)$. This implies additional symmetries for R. In particular:

- $R'(X,Y,W,Z) = R'(JX,JY,W,Z) = R'(X,Y,JW,JZ)$,
- $\mathrm{Ric}(X,Y) = \mathrm{Ric}(JX,JY)$.

A better second variation formula. Recall that a n-dimensional submanifold Σ in a $2n$-dimensional symplectic manifold (M,ω) is Lagrangian iff $\omega_{|T\Sigma} = 0$. In the Kähler context this is equivalent to the condition that

$$T\Sigma^\perp \to \Lambda^1(\Sigma), \quad Z \mapsto \zeta := \omega(Z,\cdot)$$

is an isomorphism. The condition $|\omega(e_1,\dots,e_n)| \leq 1$ shows that Lagrangian submanifolds live on the opposite side of the spectrum, compared to complex submanifolds. In this sense they are usually not calibrated. In order to understand their stability, it is thus necessary to return to the variation formulae.

Remark. There is an important exception to the above. When M is Calabi-Yau, ie Kähler with a complex volume form $\Omega \in \Lambda^{n,0}(M)$ such that $\nabla\Omega = 0$, we can define the class of special Lagrangian submanifolds by imposing the additional condition $\mathrm{Im}(\Omega)_{|T\Sigma} = 0$. These submanifolds turn out to be calibrated by $\alpha := \mathrm{Re}(\Omega) \in \Lambda^n(M)$, so calibrations methods apply.

It turns out that, in this context, rearranging the terms in the standard second variation formula (alternative version, seen above), leads to yet another expression for this formula. This expression is much better than the original one for two reasons: (i) it makes use of Ric, rather than of R, (ii) the remaining term is always non-negative.

Theorem 5.1 (Oh, 1990)**.** *Assume $\phi : \Sigma \to M$ is minimal and Lagrangian in Kähler M. Consider any normal variation Z. Then*

$$\frac{d^2}{dt^2}(\mathrm{Vol}\circ\phi)(\tau)_{|\tau=0} = \int_\Sigma (\Delta\zeta,\zeta) - \mathrm{Ric}(Z,Z)\,\mathrm{vol}_g,$$

where Δ denotes the Hodge Laplacian on Σ and Ric *is the ambient Ricci curvature. In particular, Σ (more precisely: ϕ) is stable if* $\mathrm{Ric} \leq 0$.

Indeed, the Lagrangian condition together with the fact $\nabla^M J = 0$ implies that $(\nabla^\Sigma JZ) = (\nabla^M JZ)^T = (J\nabla^M Z)^T = J(\nabla^\perp Z)$. The Weitzenböck identity, applied to $\zeta = g(JZ,\cdot)$ on Σ, then leads to $\Delta\zeta = -\omega(\mathrm{tr}_g(\nabla^\perp\nabla^\perp Z),\cdot) + \mathrm{Ric}_\Sigma(JZ,\cdot)$. Evaluating this on JZ we find

$$\begin{aligned} -\mathrm{tr}_g(\nabla^\perp\nabla^\perp Z)\cdot Z &= \Delta\zeta\cdot\zeta - \mathrm{Ric}_\Sigma(JZ,JZ) \\ &= \Delta\zeta\cdot\zeta - R_\Sigma(e_i,JZ)JZ\cdot e_i. \end{aligned}$$

The Gauss equation for curvature, together with $\nabla^M J = 0$ and $H = 0$, yields

$$\begin{aligned} R(e_i,JZ)JZ\cdot e_i &= R_\Sigma(e_i,JZ)JZ\cdot e_i + |(\nabla_{e_i}JZ)^\perp|^2 + (\nabla_{e_i}e_i)^\perp\cdot(\nabla_{JZ}JZ)^\perp \\ &= R_\Sigma(e_i,JZ)JZ\cdot e_i + |(\nabla_{e_i}Z)^T|^2. \end{aligned}$$

The Kähler condition implies that $R(e_i, JZ)JZ \cdot e_i = R(Je_i, Z)Z \cdot Je_i$. Comparing this with the second variation formula allows us to complete the curvature term so as to obtain $-\operatorname{Ric}(Z, Z)$ and to cancel the term depending on the second fundamental form.

The bigger picture. The Weitzenböck formula relates three different quantities. The Hodge Laplacian can be related to the topology of M: Hodge theory (on compact M) shows that the dimension of the space of harmonic k-forms coincides with $b_k(M)$. The covariant Laplacian can be related to linear algebra: parallel k-forms are defined by their value in one point, so the dimension of this space is $\binom{n}{k}$. The Ricci tensor is a form of curvature. We can use the identity to understand relations between these quantities. Assume, for example, that M is compact and $\operatorname{Ric} \geq 0$. Assume ξ is harmonic. A calculation then shows that $\int_M \Delta\xi \cdot \xi \operatorname{vol}_g = \int_M \nabla\xi \cdot \nabla\xi \operatorname{vol}_g + \int_M \operatorname{Ric}(X, X) \operatorname{vol}_g$, so $\nabla\xi = 0$. This implies a bound on topology: $b_1(M) \leq \dim(T_p^*M) = \dim(M)$.

Analogous formulae exist for k-forms. Non-negative curvature then leads to bounds $b_k(M) \leq \binom{n}{k}$.

Arguments of this type, applied to various operators, generally go under the name of Bochner technique: they can be used to study harmonic functions, Killing vector fields, spinors, etc. We refer to Petersen, Riemannian Geometry, for details.

6 Coassociative submanifolds

Toolbox. We do not have time to discuss G_2 geometry in the detail needed to prove the results below. The calculations mostly rely on formulae developed in Bryant, "Some remarks on G_2 structures". In these notes we follow his same conventions. Another useful reference is Karigiannis, "Introduction to G_2 geometry", which however follows different conventions.

A Kähler-G_2 dictionary. The above presentation of Kähler and G_2 manifolds makes clear that, beyond the obvious differences, there are certain formal analogies between them. This is clarified by the following dictionary:

$$\omega \simeq \varphi, \quad \text{cpx subs} \simeq \text{associative subs.}$$

The Lagrangian condition $\omega_{|T\Sigma} = 0$ corresponds to the condition $\varphi_{|T\Sigma} = 0$. The maximum possible dimension for this condition is $\dim(\Sigma) = 4$. Such submanifolds are known as coassociative.

The pointwise models for associatives and coassociatives are, respectively, $\mathbb{R}^3$ and $\mathbb{R}^4$ in $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$. Notice, in our model 3-form φ_{std}, the appearance of the std self-dual 2-forms on $\mathbb{R}^4$: $e^{45} + e^{67}$, $e^{46} - e^{57}$, $e^{56} + e^{47}$. This implies that, if $Z \in \mathbb{R}^3 = (\mathbb{R}^4)^\perp$, then $\varphi_{std}(Z, \cdot, \cdot) \in \Lambda^2_+(\mathbb{R}^4)$.

Our dictionary thus continues as follows: the isomorphism $Z \in T\Sigma^\perp \simeq \zeta = \omega(Z, \cdot) \in \Lambda^1(\Sigma)$ for Lagrangians corresponds to an isomorphism $Z \in T\Sigma^\perp \simeq \varphi(Z, \cdot, \cdot) \in \Lambda^2_+(\Sigma)$ for coassociatives.

A better second variation formula. The above leads us to the question whether there exists a better second variation formula for coassociative submanifolds, similar to Oh's formula for Lagrangians in Kähler.

Figuring out how to rearrange the terms in the standard second variation formula to take advantage of the features of G_2 geometry requires some effort. The end result is that there seems not to exist a general analogue of the Lagrangian formula. Coassociative submanifolds have however two special features:

- They are calibrated by $\psi := \star\varphi \in \Lambda^4(M)$. This implies that ψ provides a global extension of vol_g, from Σ to M, encouraging us to try to extend other elements of the geometry of Σ to M.

  We shall not assume, however, that $d\psi = 0$, so our coassociatives are not automatically minimal/minimizing.

- Coassociative submanifolds generate smooth, finite-dimensional, moduli spaces $\mathcal{M}$.

G_2 theory shows that there exists a 2-form τ_2 on M such that $d\psi = \tau_2 \wedge \varphi$. We can now state alternative first and second variation formulae for coassociative submanifolds. They are proved in Pacini, "Variation formulae for the volume of coassociative submanifolds".

Proposition 6.1. *Let Σ be coassociative and Z be an infinitesimal deformation. Then*

$$\frac{d}{d\tau}(\mathrm{Vol} \circ \phi)_{|\tau=0} = \int_\Sigma \tau^+_{2|\Sigma} \wedge \iota_{Z^\perp}\phi.$$

In other words, $-H$ can be identified with the self-dual component of the restriction of τ_2: $-\iota_H \phi_{|\Sigma} = \tau^+_{2|\Sigma}$. In particular, $H = 0$ iff $\tau_{2|\Sigma} \in \Lambda^2_-(\Sigma)$.

Theorem 6.2 (Pacini, 2022)**.** *Let Σ be a minimal compact coassociative submanifold in M. Let $\mathcal{M}$ denote the moduli space of its coassociative deformations. Consider the standard volume functional* Vol*, restricted to $\mathcal{M}$.*

Choose any variation $\phi(\tau)$ in $\mathcal{M}$. We may assume the corresponding infinitesimal variation Z is normal. Then

$$\frac{d^2}{d\tau^2}(\mathrm{Vol} \circ \phi)_{|\tau=0} = \int_\Sigma \tau_2 \wedge \gamma_Z - \mathrm{Ric}(Z, Z)\,\mathrm{vol}_g,$$

where $\gamma_Z \in \Lambda^2_-(\Sigma_0)$ is defined in terms of the second fundamental form:

$$\gamma_Z(X_1, X_2) := \iota_Z\phi((\nabla_{X_1} Z)^T, X_2) + \iota_Z\phi(X_1, (\nabla_{X_2} Z)^T).$$