Affine and Projective Geometry

Adrien Deloro

Şirince ’22 Summer School

These are lecture notes with exercises for a two week course of $12 \times 2 = 24$ hours given in 2016, 2107, and 2022 at the Nesin Matematik Köyü in Şirince, Turkey. They should be accessible to a 3rd year student in mathematics; syllabus and prerequisites are described below. I wish to thank Joshua Wiscons for useful conversations, and Fırat Kıyak for his many questions and suggestions.

If found with flaws or mistakes, this document compiled on 9th August 2022 should be returned to adrien.deloro@imj-prg.fr. Please mention the date.

Contents

Each section corresponds to a lecture of 2 hours, with the introduction fitting into § 1. The class can be taught in a different order as there are independences between lectures. §§ 1–6 form the initial socle. Then §§ 7–8 are one block; so are §§ 10–11. Apart from these dependencies, lectures § 7–11 can be permuted freely.

Introduction

The real plane $\mathbb{R}^2$ may be studied with different notions and tools; these are called *structural layers* and are conveniently imagined as *tracing sheets*. The following list is not comprehensive, and items may be combined or not:

- points and lines;
- coordinates and vectors;
- distances;

- areas;
- the dot product;
- the complex multiplication;
- angles;
- angle measurements.

These lead to distinct mathematical theories, all developed in order to formalise our intuition of what 'the physical plane' is.

In this course we restrict ourselves to the most basic layer where one has only points and lines. The relevant notion is that of an *affine plane*. It is just a collection of points and lines; a point can be on a line or not. Geometric intuition, or physical observation in our small part of the universe (a sheet of paper can do) will suggest axioms describing the behaviour. These axioms define *abstract affine planes*, and the usual affine plane $\mathbb{R}^2$ is only one example of such structures. Quickly one realises that *projective* geometry behaves in a nicer way. Essentially it amounts to adding 'points at infinity' to an affine plane. The resulting *abstract projective planes* are more homogeneous, more worth a mathematician's time. The course is dedicated to their study.

When compared to analytic number theory, differential geometry or algebraic topology, our topic is certainly elementary since there is no 'higher' structure (no topologies, no categories, etc.). But the word *elementary* is no synonym of *easy*: it refers to a set of methods, not to a level of difficulty. Incidence geometries were studied, among others, by Hilbert, E. Artin, Tarski; and are therefore not an unworthy subject. Beyond the first study of abstract projective planes lies *combinatorial geometry*, full of challenging open problems. We will not go that far. Further reading is suggested at the end of the lecture notes.

The course covers basic material on projective planes. It describes beautiful phenomena. It returns to elementary aspects and therefore teaches one to separate structural layers. It exemplifies the power of group-theoretic language. And it is not part of the standard curriculum. For all these reasons it is very suitable for a summer school.

Prerequisites

The class is supposedly accessible to a third year student.

Geometry: High-school level. The use of cartesian coordinates; line equations, the difference between points and vectors.

Linear algebra: Basic knowledge. Vector spaces and coordinates. We shall need the (lesser known) basic linear algebra over skew-fields, which works exactly the same however confusing it might sound at first.

Fields and skew-fields: The definitions. One should be aware of the existence of the latter as they appear everywhere in the course.

Group theory: Groups will play a prominent role in advanced sections like §§ 7, 8, 9, 10, 11. To follow them one should be fully comfortable with subgroups, normal subgroups, and group actions (stabilisers, transitivity).

Before we start, recall two definitions and related phenomena.

Fields and skew-fields; Wedderburn's theorem

Definition (field). A *field* is a structure $(\mathbb{F}; 0, 1, +, \cdot)$ where:

- $(\mathbb{F}; 0, +)$ is an abelian group (called 'the additive group');
- $(\mathbb{F} \setminus \{0\}; 1, \cdot)$ is an abelian group (called 'the multiplicative group');
- $\cdot$ is distributive over $+$, viz. $(a + b) \cdot c = a \cdot c + b \cdot c$ and likewise on the other side.

Equipped with their natural algebraic operations, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. One should know that there exist *finite fields*.

Theorem (Galois). *For every prime power $q = p^k$, there exists a (unique up to isomorphism) field with q elements. Every finite field has order a prime power.*

One should also know that commutativity may be dropped.

Definition (skew-field). A *skew-field* is a structure $(\mathbb{F}; 0, 1, +, \cdot)$ where all axioms of a field hold, *except that $\cdot$ is not required to be commutative.*

An example of a non-commutative skew-field is the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$, subject to the usual 'Hamilton identities'. There one has $ij = k \neq -k = ji$. It should be checked or known that $\mathbb{H}$ is a skew-field, but not a field. For reference one may see my 2021 course on quaternions.[1]

Theorem (Wedderburn). *Any* finite *skew-field is commutative.*

We shall not prove this classical number-theoretic result but it considerably clarifies the picture. Though non-commutative skew-fields are therefore uncommon objects in general mathematics, they play a central, unavoidable role in the study of affine and projective planes. This involves doing some linar algebra over skew-fields.

Remark (linear algebra over a skew-field). One has to distinguish between *left*- and *right*-vector spaces, not confusing them. (The risk is serious when $V = \mathbb{F}^n$.) Basic linear algebra such as elementary elimination theory and dimension theory still apply.

On the other hand, there is no fully satisfactory/fully accepted theory of determinants over a skew-field. Therefore claiming invertibility of a matrix with entries in a non-commutative skew-field because of its 'determinant' is simple nonsense.

1 Affine planes

Abstract. § 1.1 formalises the setting: *incidence geometries*. In common cases, the abstract incidence relation can be taken to be set-theoretic membership. § 1.2 introduces *affine planes*, and the notion of *parallelism*. § 1.3 associates to any skew-field $\mathbb{F}$ a concrete affine plane $\mathbb{A}^2(\mathbb{F})$, which is the usual system of points and lines in $\mathbb{F}^2$. Not all affine planes are of this form.

[1] https://webusers.imj-prg.fr/~adrien.deloro/teaching-archive/Sirince-Quaternions.pdf

1.1 Incidence geometries

The purpose of this course is a description of certain point-line configurations. There are two natural approaches here:

- work with points only and a ternary collinearity relation (with intended meaning: 'the three points are on some common line');
- work with points *and* lines as objects of distinct types, and a binary incidence relation (with intended meaning: 'the point is on the line').

Though the former option has its virtues, we retain the latter for better readibility.

1.1.1. Definition (incidence geometry). An *incidence geometry* is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ where:

- $\mathcal{P}$ is a set of objects called 'points' ;
- $\mathcal{L}$ is a set of objects called 'lines';
- I is a relation on $\mathcal{P} \times \mathcal{L}$ called 'incidence', which intuitively says when a point is on a line or not.

Manipulating two distinct sets of objects requires using variables in a consistent manner. From now on, $a, b, \dots,\ a', b_1, \dots,\ p, q, \dots$ will always denote points, while ℓ, m, ℓ', $m_1, \dots$ will always denote lines. This is implicit in our notation. For instance, $(\forall a)$ means: 'for any *point* $a\dots$'.

1.1.2. Definition (collinear). Given an incidence geometry, points $a_1, \dots, a_n$ are *collinear* if there is a line ℓ such that for each k one has $a_k I \ell$.

The incidence relation I is of an abstract nature; instead of 'p is incident to ℓ' one may be more used to writing '$p \in \ell'$. Indeed replacing I by $\in$ is allowed, up to isomorphism. This requires a definition.

1.1.3. Definition (isomorphism). Two incidence geometries $\Gamma_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1)$ and $\Gamma_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2)$ are *isomorphic* if there are bijections $f: \mathcal{P}_1 \to \mathcal{P}_2$ and $g: \mathcal{L}_1 \to \mathcal{L}_2$ such that:

$$(\forall a)(\forall \ell)(a I_1 \ell \leftrightarrow f(a) I_2 g(\ell)).$$

1.1.4. Proposition (reducing to membership). *Let $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ be an incidence geometry such that any two lines having the same points are actually equal. Then there is an incidence geometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \in)$ where incidence is set-theoretic membership, and $\Gamma \simeq \Gamma'$ as geometries.*

Proof. The idea is to keep the same points, and write new lines as sets of points. To each $\ell \in \mathcal{L}$ associate $S_\ell = \{p \in \mathcal{P} : p I \ell\}$, a subset of $\mathcal{P}$. Now consider $\mathcal{L}' = \{S_\ell : \ell \in \mathcal{L}\}$, a family of subsets of $\mathcal{P}$. We claim that $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \in)$ are isomorphic.

Let f be the identity function on $\mathcal{P}$ and $g: \mathcal{L} \to \mathcal{L}'$ map ℓ to S_ℓ. Clearly f is a bijection and g is surjective. Actually g is injective: if $S_\ell = S_m$ then lines ℓ and m have the same points, so by assumption they are equal. Finally for any $(p, \ell) \in \mathcal{P} \times \mathcal{L}$, one has $p I \ell$ iff $p \in S_\ell$. Hence (f, g) is an isomorphism $\Gamma \simeq \Gamma'$. □

So instead of always saying that 'p is incident to ℓ' in Γ one may implicitly work in Γ' and use the common phrases: 'p is in ℓ', 'p belongs to ℓ', 'ℓ contains p'.

1.2 The definition

The definition of an affine plane (Definition 1.2.2 below) requires an auxiliary notion.

1.2.1. Definition (parallel). Two lines ℓ, m are *parallel* if either $\ell = m$ or there is no point incident to both, viz.:

$$\ell /\!/ m \quad \text{iff} \quad [\ell = m] \vee [(\forall a)\neg(aI\ell \wedge aIm)].$$

(Notice that it captures the intuition of $\mathbb{R}^2$, not of $\mathbb{R}^3$.) This notion plays a role only when studying affine planes, and disappears when studying projective planes (§ 2).

1.2.2. Definition (affine plane). An *affine plane* $\mathbb{A}$ is an incidence geometry satisfying the following axioms $\mathbf{AP_1}$, $\mathbf{AP_2}$, $\mathbf{AP_3}$.

$\mathbf{AP_1}$. $(\forall a)(\forall b)[(a \neq b) \to (\exists!\ell)(aI\ell \wedge bI\ell)]$.

(Meaning: given any two distinct points, there is a unique line incident to both.)

$\mathbf{AP_2}$. $(\forall a)(\forall \ell)(\exists! m)[(aIm) \wedge (\ell /\!/ m)]$.

(Meaning: through any point there is a unique line parallel to a given line.)

$\mathbf{AP_3}$. There exist three non-collinear points.

1.2.3. Remarks.

- The axiomatisation is in the sole terms of points and lines; words such as *angle*, *distance*, or even *between* are banned from the vocabulary. They would lead to other (more powerful and less elementary, but equally interesting) mathematical theories.
- $\mathbf{AP_3}$ removes degenerate configurations, the most extreme of which being $\mathcal{P} = \varnothing = \mathcal{L}$. See exercice 1.4.3.
- By the axioms and proposition 1.1.4, one may suppose that the incidence relation is $\in$. See exercise 1.4.4.

1.3 The affine plane over a skew-field

We wrote the definition of an affine plane by 'painting from nature', viz. by observing the real affine plane. Our definition captures more than the real case; skew-fields appear naturally.

- If $\mathbb{F}$ is a skew-field, then elements of $\mathbb{F}^2$ will be *written as rows, not as columns*, viz. as pairs (x, y).
- We use the same pair notation for points and vectors; this results in no confusion. Recall the following basics of high-school affine geometry.
- Vectors may be added by putting $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.
- Scalars act on vectors by letting $\lambda \cdot (x, y) = (\lambda \cdot x, \lambda \cdot y)$; this turns $\mathbb{F}^2$ into a *left*-$\mathbb{F}$-vector space.
- Last, vectors act on points: if $p = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$, let $p + \vec{v} = (x_1 + x_2, y_1 + y_2)$. (This may look like the sum of two vectors, but here p is a point, and there is no addition of points. Our notation simply does not reflect the essential difference in nature between points and vectors.)

1.3.1. Definition (affine plane over a skew-field $\mathbb{A}^2(\mathbb{F})$). Let $\mathbb{F}$ be a skew-field. Let $\mathbb{A}^2(\mathbb{F})$ be the following incidence geometry:

- points are pairs $(x, y) \in \mathbb{F}^2$;
- lines are sets of the form $\{p + t\vec{v} : t \in \mathbb{F}\}$ for p a point and $\vec{v} \neq \vec{0}$ a non-zero vector;
- incidence is set-theoretic membership.

This generalises the familiar case of $\mathbb{A}^2(\mathbb{R})$ as it allows skew-fields. One should practice a little with skew-fields (using quaternions for instance).

1.3.2. Proposition. *Let $\mathbb{F}$ be a skew-field. Then $\mathbb{A}^2(\mathbb{F})$ is an affine plane.*

1.3.3. Remark (important). *Not all affine planes are of the form $\mathbb{A}^2(\mathbb{F})$.* The Desargues property (§ 3) characterises those affine planes of the form $\mathbb{A}^2(\mathbb{F})$.

Proof. Points are pairs $p = (x, y) \in \mathbb{F}^2$; lines are sets of the form

$$\ell_{p,\vec{v}} = \{p + t\vec{v} : t \in \mathbb{F}\}$$

for $p \in \mathbb{F}^2$ and $\vec{v} \neq \vec{0}$. Notice that one always has $p \in \ell_{p,\vec{v}}$.

We must check axioms $\mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}$. Call two vectors *proportional* if both are non-zero and one (equivalently, each) is a multiple of the other. (Since the field is non-commutative, we do mean $\vec{v_2} = \lambda \cdot \vec{v_1}$ with λ on the left. For a left-vector space we never write 'vector times scalar'.)

Step 1. Two lines $\ell_1 = \ell_{p_1,\vec{v_1}}$ and $\ell_2 = \ell_{p_2,\vec{v_2}}$ are equal iff [$\vec{v_1}$ and $\vec{v_2}$ are proportional vectors, and $\overrightarrow{p_1p_2}$ is a (possibly null) multiple of those].

Verification. First suppose $\ell_1 = \ell_2$. Since $p_2 \in \ell_1$ there is $t \in \mathbb{F}$ with $p_2 = p_1 + t\vec{v_1}$. So $\overrightarrow{p_1p_2} = t\vec{v_1}$ is a multiple of $\vec{v_1}$. But also $p_1 \in \ell_2$ so there is $s \in \mathbb{F}$ with $p_1 = p_2 + s\vec{v_2}$. Now:

$$p_1 = p_2 + s\vec{v_2} = p_1 + t\vec{v_1} + s\vec{v_2},$$

which simplifies into $t\vec{v_1} + s\vec{v_2} = \vec{0}$. Since neither $\vec{v_1}$ nor $\vec{v_2}$ is $\vec{0}$, they are proportional.

Conversely we shall prove one inclusion, and the other will follow by symmetry. So let $q \in \ell_1$. By definition there is $t \in \mathbb{F}$ with $q = p_1 + t\vec{v_1}$. Now by hypothesis there are $\lambda, \mu \in \mathbb{F}$ with $\overrightarrow{p_1p_2} = \lambda\vec{v_2}$ and $\vec{v_1} = \mu\vec{v_2}$ (we even know that $\mu \neq 0$). Then:

$$q = p_2 - \overrightarrow{p_1p_2} + t\vec{v_1} = p_2 + (t\mu - \lambda)\vec{v_2} \in \ell_2,$$

proving $\ell_1 \subseteq \ell_2$. ◇

Step 2. Two lines $\ell_1 = \ell_{p_1,\vec{v_1}}$ and $\ell_2 = \ell_{p_2,\vec{v_2}}$ are parallel iff $\vec{v_1}$ and $\vec{v_2}$ are proportional vectors.

Verification. Suppose $\ell_1 /\!/ \ell_2$; we wish to show that $\vec{v_1}$ and $\vec{v_2}$ are proportional vectors. If $\ell_1 = \ell_2$ then this follows from the previous claim. So suppose $\ell_1 \neq \ell_2$; since they are parallel this means that there is no point on both lines. Hence the equation:

$$p_1 + t_1\overrightarrow{v_1} = p_2 + t_2\overrightarrow{v_2}$$

has *no* solution for $(t_1, t_2) \in \mathbb{F}^2$.

Work in coordinates, say $p_i = (x_i, y_i)$ and $\overrightarrow{v_i} = (a_i, b_i)$. So the system of equations:

$$(\mathcal{S}): \quad \begin{cases} x_1 + t_1 a_1 & = & x_2 + t_2 a_2 \\ y_1 + t_1 b_1 & = & y_2 + t_2 b_2 \end{cases}$$

has no solution for $(t_1, t_2) \in \mathbb{F}^2$. Notice that multiplicative coefficients are on the right and variables t_1, t_2 on the left; this is due to non-commutativity of $\mathbb{F}$.

Remember that we want to prove that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are proportional vectors, and that the assumption is that $(\mathcal{S})$ has no solution. There are three cases, two of which are equivalent.

Case 1a. Suppose $a_1 = 0$. Then since $\overrightarrow{v_1} \neq \overrightarrow{0}$ one has $b_1 \neq 0$. If $a_2 \neq 0$ then there is a unique solution $t_2 = (x_1 - x_2) \cdot a_2^{-1}$ to the first equation; now t_2 being known there is a unique solution in t_1 to the second. This is a contradiction to $(\mathcal{S})$ having no solution, so actually $a_2 = 0$. Then since $\overrightarrow{v_2} \neq \overrightarrow{0}$ one has $b_2 \neq 0$, and clearly $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are proportional vectors.

Case 1b. If $b_1 = 0$, the desired conclusion follows similarly.

Case 2. Suppose $a_1 \neq 0$ and $b_1 \neq 0$. Then multiply the first equation *on the right* by $\lambda = a_1^{-1} b_1$, and subtract from the second, getting:

$$y_1 - x_1\lambda = y_2 - x_2\lambda + t_2(b_2 - a_2\lambda).$$

If $b_2 - a_2\lambda \neq 0$ then there is a (unique) solution in t_2, which then gives rise to a (unique) solution in t_1: so $(\mathcal{S})$ has a solution, a contradiction.

Therefore $b_2 = a_2\lambda = a_2 a_1^{-1} b_1$. Finally:

$$\overrightarrow{v_2} = (a_2, a_2 a_1^{-1} b_1) = a_2 a_1^{-1} \cdot (a_1, b_1) = a_2 a_1^{-1} \overrightarrow{v_1},$$

so $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are proportional vectors. ◇

Step 3. $\mathbb{A}^2(\mathbb{R})$ satisfies axioms $\mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}$.

Verification. We check them in order.

$\mathbf{AP_1}$. Let $p_1 \neq p_2$ be points. Then $\overrightarrow{v} = \overrightarrow{p_1 p_2} \neq \overrightarrow{0}$ and $\ell = \ell_{p_1, \overrightarrow{v}}$ clearly contains p_1 and p_2. Uniqueness is clear as well.

$\mathbf{AP_2}$. Let p be a point and $\ell = \ell_{q, \overrightarrow{v}}$ be a line. Then clearly $m = \ell_{p, \overrightarrow{v}}$ contains p and is parallel to ℓ; but also uniqueness is clear.

$\mathbf{AP_3}$. Simply take $(0, 0)$, $(0, 1)$, and $(1, 0)$. ◇

This completes the proof. □

1.3.4. Remark (the right-version $(\mathbb{F})^2\mathbb{A}$).

- Using the *right* action, define a geometry $(\mathbb{F})^2\mathbb{A}$ with 'right-lines' $\{p+\overrightarrow{v}t : t \in \mathbb{F}\}$.
- Points in $\mathbb{A}^2(\mathbb{F})$ and $(\mathbb{F})^2\mathbb{A}$ are the same, but lines may differ. Technically, sets $\{p+t\overrightarrow{v} : t \in \mathbb{F}\}$ and $\{p+\overrightarrow{v}t : t \in \mathbb{F}\}$ need not be equal. (This may seem confusing at first, but the left- and right-linear structures are simply not the same.)
- $(\mathbb{F})^2\mathbb{A}$ is another affine plane: same proof as Proposition 1.3.2, exchanging sides.
- For a *commutative* field $\mathbb{F}$, left- and right-lines are equal, so $\mathbb{A}^2(\mathbb{F}) = (\mathbb{F})^2\mathbb{A}$. For quaternions $\mathbb{H}$, lines differ but $\mathbb{A}^2(\mathbb{H}) \simeq (\mathbb{H})^2\mathbb{A}$ (exercise 1.4.7). For arbitrary $\mathbb{F}$, planes $\mathbb{A}^2(\mathbb{F})$ and $(\mathbb{F})^2\mathbb{A}$ need not be isomorphic.

1.4 Exercises

1.4.1. Exercise.

1. *Write* $\mathbf{AP_3}$ *in logical form.*
2. *(Tedious.) Using* only points *and a ternary relation for collinearity, define parallelism, then axiomatise affine planes.*

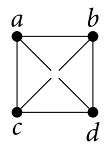
1.4.2. Exercise. *In an affine plane, prove that* $\parallel$ *is an equivalence relation.*

1.4.3. Exercise. *List all incidence geometries satisfying* $\mathbf{AP_1}$ *and* $\mathbf{AP_2}$ *but not* $\mathbf{AP_3}$.

1.4.4. Exercise. *Let* $\mathbb{A} = (\mathcal{P}, \mathcal{L}, I)$ *be an affine plane. Prove that any line has at least two points, and that two lines with the same points are equal. Conclude that there* $\mathbb{A}$ *is isomorphic to an affine plane of the form* $\mathbb{A}' = (\mathcal{P}', \mathcal{L}', \in)$.

1.4.5. Exercise. *Let* $\mathcal{P}$ *be the set of points on the Euclidean sphere* $\mathbb{S}^2(\mathbb{R}) = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ *and* $\mathcal{L}$ *be the set of 'great circles', viz. circles of maximum radius on the sphere. Prove that* $(\mathcal{P}, \mathcal{L}, \in)$ *is a geometry satisyfing* $\mathbf{AP_1}$. *What about* $\mathbf{AP_2}$*?*

1.4.6. Exercise. *Prove that the following is the smallest affine plane. Can you identify it?*



One point is removed;
diagonals do not meet.

1.4.7. Exercise. *Let* $\mathbb{F}$ *be a skew-field.*

1. *Check that* $(\mathbb{F})^2\mathbb{A}$ *of Remark 1.3.4 is an affine plane.*
2. *Prove that* $(\mathbb{F})^2\mathbb{A} = \mathbb{A}^2(\mathbb{F})$ *iff* $\mathbb{F}$ *is commutative.*
3. *Suppose that there is an* anti-automorphism *of* $\mathbb{F}$*, viz. an additive bijection* φ *such that* $(\forall a, b)(\varphi(a \cdot b) = \varphi(b) \cdot \varphi(a))$. *Show that* $(\mathbb{F})^2\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$. *(The converse can be proved in later exercises.)*

2 Projective planes

Abstract. § 2.1 defines *projective planes* and gives one class of examples: the structures $\mathbb{P}^2(\mathbb{F})$ for $\mathbb{F}$ a skew-field. Not all projective planes are of this form. § 2.2 explores the relationships between the classes of abstract affine planes and of abstract projective planes by so-called *projectivisation* and *affinisation*. § 2.3 applies these operations to the concrete examples $\mathbb{A}^2(\mathbb{F})$ and $\mathbb{P}^2(\mathbb{F})$.

The need to 'add points at infinity' where parallel lines would meet was felt during the Renaissance, and cannot be separated from the history of Painting. Early school pencil-and-paper practice shapes our first intuition of geometry, and is convincingly rendered by affine planes. However, trying to represent on a planar support the physical 3-dimensional space, begs for other means than linear projections. Indeed, our seeing device (the eye) is a nearly punctual object. The theory of perspective thus naturally takes place in projective geometry.

Mathematically, projective geometry is even more natural than its affine version. Statements there are more concise; the algebraic behaviour is more homogeneous. This course is really about projective planes.

2.1 The definition

2.1.1. Definition (projective plane)**.** A *projective plane* is an incidence geometry $(\mathcal{P}, \mathcal{L}, I)$ satisfying the following axioms $\mathbf{PP_1}, \mathbf{PP_2}, \mathbf{PP_3}$.

$\mathbf{PP_1}$. Through any two distinct points there is a unique line.

$\mathbf{PP_2}$. Any two distinct lines meet in a unique point.

$\mathbf{PP_3}$. There are four points no three of which are collinear.

(The definition has a lovely symmetry called self-duality, to which we return in § 12.)

2.1.2. Remark (and notation)**.** Both $\mathbf{AP_1}$ and $\mathbf{PP_1}$ say that any two distinct points lie on a unique line. From now on, given two points $a \neq b$ of an affine or projective plane, (ab) denotes the unique line through them.

We move to one source of examples. In a left-vector space, a *(left-)vector line* is a 1-dimensional left-vector subspace, which can be viewed as a line through the origin; define a *(left-)vector plane* likewise.

2.1.3. Definition (projective plane over a skew-field $\mathbb{P}^2(\mathbb{F})$)**.** Let $\mathbb{F}$ be a skew-field. Let $\mathbb{P}^2(\mathbb{F})$ be the following incidence geometry:

- as points, the left-vector lines of $\mathbb{F}^3$;
- as points, the left-vector planes of $\mathbb{F}^3$;
- as incidence relation, set-theoretic inclusion $\subseteq$.

2.1.4. Remark. There also exists a *right*-version $(\mathbb{F})^2\mathbb{P}$, which need not be equal *nor even isomorphic* to $\mathbb{P}^2(\mathbb{F})$ (see Remark 1.3.4).

2.1.5. Proposition. *Let $\mathbb{F}$ be a skew-field. Then $\mathbb{P}^2(\mathbb{F})$ is a projective plane.*

2.1.6. Remark (important). Like in the affine case, *not all projective planes are of the form $\mathbb{A}^2(\mathbb{F})$*. The Desargues property (§ 3) characterises those projective planes of the form $\mathbb{P}^2(\mathbb{F})$.

2.1.7. Remark (continued). For instance, $\mathbb{P}^2(\mathbb{F}_9)$ has $9^2 + 9 + 1 = 91$ points. But there exist *three more* projective planes with 91 elements. To show that in total there are exactly four, requires heavy computer assistance.[2]

Proof. We must check three things. This is easy as our construction relies on vectors in dimension 3. Even though $\mathbb{F}$ need not be commutative, basic linear algebra remains sound. We only use left-subspaces, and consequently omit 'left'.

PP$_1$. Let $\alpha \neq \beta$ be two projective points, i.e. two distinct vector lines $L_\alpha, L_\beta \subseteq \mathbb{F}^3$. Then set $H = \langle L_\alpha, L_\beta \rangle \subseteq \mathbb{F}^3$, a vector plane of $\mathbb{F}^3$. In projective terms, H is a line λ. Now by definition, $L_\alpha \subseteq H$ translates into $\alpha I \lambda$, and $\beta I \lambda$ likewise. Uniqueness is clear as well, since there is no other vector plane containing both L_α and L_β.

PP$_2$. Start with two distinct projective lines $\lambda \neq \mu$. By construction they correspond to distinct vector planes $H_\lambda \neq H_\mu$, and $H_\lambda \cap H_\mu$ is a vector line L, which we see as a projective point α. Obviously $\alpha I \lambda$ and $\alpha I \mu$, but uniqueness is clear as well.

PP$_3$. Vector lines spanned by vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ define suitable projective points. □

The remarkable simplicity of the proof (cf. proof of Proposition 1.3.2) is already an indication in favour of projective over affine geometry: even for 'coordinatisable' objects, the projective version is easier to deal with.

2.1.8. Remark (projective coordinates). One can represent points of $\mathbb{P}^2(\mathbb{F})$ by equivalence classes of non-zero triples of coordinates, as follows. Consider the relation on $\mathbb{F}^3 \setminus \{(0,0,0)\}$:

$$(a_1, a_2, a_3) \sim (b_2, b_2, b_3) \quad \text{iff} \quad (\exists \lambda)[(\lambda a_1 = b_1) \wedge (\lambda a_2 = b_2) \wedge (\lambda a_3 = c_3)].$$

This is the equivalence relation of *proportionality* of non-zero vectors. The set of equivalence classes is naturally the set of vector lines in $\mathbb{F}^3$, i.e. points of $\mathbb{P}^2(\mathbb{F})$.

Write $[a_1, a_2, a_3]$ for the class of (a_1, a_2, a_3); they are *projective coordinates*, defined up to (left-)multiplication by a non-zero scalar. If $[a_1, a_2, a_3] = [b_1, b_2, b_3]$ and $a_1 = b_1 \neq 0$, then $a_2 = b_2$ and $a_3 = b_3$.

Projective coordinates are useful when bringing matrices into the picture. However, since we treat $\mathbb{F}^3$ as a *left*-vector space, *matrices will act from the right*, viz. by row-matrix product.

2.2 From affine to projective, and back

Another way to obtain projective planes is by 'completing' affine planes, a procedure we now describe. In the affine world, some intersections are missing: if $\ell \parallel m$ are distinct, parallel affine lines, then we should add a 'point at infinity' where they meet at last. Now

[2] C. Lam, G. Kolesova, L. Thiel. A computer search for finite projective planes of order 9, *Discr. Math.*, 92, pp.187–195, 1991.

if $\ell \mathbin{/\!/} m \mathbin{/\!/} n$ then the *same* point should be added for the missing intersection $\ell \cap m$ and for the missing intersection $m \cap n$. So one is not working with pairs of parallel lines, but with whole sets of pairwise parallel lines.

2.2.1. Lemma. *Let $\mathbb{A}$ be any affine plane. Then $/\!/$ is an equivalence relation.*

Proof. This holds of $\mathbb{A}^2(\mathbb{F})$ for $\mathbb{F}$ a skew-field, as one has a detailed description of parallelism from the proof of Proposition 1.3.2. But we want a *general* proof, a proof using only axioms $\mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}$. Remember that in this abstract setting ℓ and m are parallel iff $\ell = m$ or ℓ and m do not meet. We freely replace $\mathbb{A}$ by an isomorphic plane where incidence is given by membership.

There are three things to check:

Reflexivity. Every line ℓ satisfies $\ell = \ell$, so $\ell \mathbin{/\!/} \ell$.

Symmetry. If $\ell \mathbin{/\!/} m$ then either $\ell = m$ or $\ell \cap m = \varnothing$; in either case $m \mathbin{/\!/} \ell$.

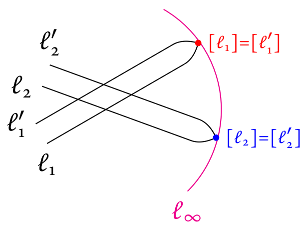
Transitivity. Suppose $\ell /\!/ m$ and $m /\!/ n$; we show that $\ell /\!/ n$. If $\ell = n$ we are done. Otherwise we must prove $\ell \cap n = \varnothing$. So suppose not; by $\mathbf{AP_1}$ there is $a \in \ell \cap n$. Now by $\mathbf{AP_2}$, there is a unique line parallel to m and containing a; however this applies to both ℓ and n. So actually $\ell = n$, a contradiction showing that $\ell \cap n = \varnothing$. Hence $\ell \mathbin{/\!/} n$, as desired. □

2.2.2. Notation. For ℓ an affine line, let $[\ell]$ be its equivalence class and call it its *direction*.

Intuitively one should add a new point 'at infinity' for each direction; there will also be a line 'at infinity'. But ordinary, affine lines ℓ are now too short: we must force a projective line to contain the direction. This explains why we go through $\hat{\ell}$ below.

2.2.3. Definition (projectivisation $\hat{\mathbb{A}}$ of an affine plane)**.** Let $\mathbb{A} = (\mathcal{P}, \mathcal{L}, \in)$ be an affine plane. The *projectivisation* of $\mathbb{A}$ is the incidence geometry $\hat{\mathbb{A}} = (\hat{\mathcal{P}}, \hat{\mathcal{L}}, \in)$ defined as follows:

- for each $\ell \in \mathcal{L}$, let $\hat{\ell} = \ell \cup \{[\ell]\}$ be the 'completion of line ℓ';
- let $\ell_\infty = \{[\ell] : \ell \in \mathcal{L}\}$ be the 'line of directions';
- $\hat{\mathcal{P}} = \mathcal{P} \cup \ell_\infty = \mathcal{P} \cup \{[\ell] : \ell \in \mathcal{L}\}$,
- $\hat{\mathcal{L}} = \{\hat{\ell} : \ell \in \mathcal{L}\} \cup \{\ell_\infty\}$.



Complete each affine line ℓ into $\hat{\ell}$.
Do not forget line at infinity ℓ_∞.

2.2.4. Proposition. *Let $\mathbb{A}$ be an affine plane. Then $\hat{\mathbb{A}}$ is a projective plane.*

Proof. There are three axioms to check.

PP₁. Let $\alpha \neq \beta$ be two points in $\hat{\mathcal{P}}$. We see several cases.

- If $\alpha = a \in \mathcal{P}$ and $\beta = b \in \mathcal{P}$, then by **AP₁** there is a unique line $\ell \in \mathcal{L}$ containing both. Clearly $\alpha, \beta \in \hat{\ell}$. We also have uniqueness. If another line $\lambda \in \hat{\mathcal{L}}$ contains α and β, then λ cannot be ℓ_∞, so $\lambda = \hat{m}$ for some affine line $m \in \mathcal{L}$. Then $a, b \in m$, so by uniqueness in **AP₁** one has $m = \ell$ and therefore $\lambda = \hat{m} = \hat{\ell}$. So $\hat{\ell}$ is the only line of $\hat{\mathcal{L}}$ containing α and β.
- Suppose $\alpha = a \in \mathcal{P}$ but $\beta \in \hat{\mathcal{P}} \setminus \mathcal{P} = \ell_\infty$ (the other case is similar). Then by definition, there is $\ell \in \mathcal{L}$ with $\beta = [\ell]$. Now by **AP₂** there is a unique $m \in \mathcal{L}$ with $a \in m$ and $m \parallel \ell$. Then on the one hand $\alpha \in \hat{m}$, and on the other hand $[\ell] = [m] \in \hat{m}$. Now to uniqueness. If a line λ contains a and β, then it cannot be ℓ_∞. So it is of the form $\hat{n}$ for some affine line n. Now $\alpha \in \hat{n}$ implies $a \in n$, and $\beta = [\ell] \in \hat{n}$ implies $[\ell] = [n]$, that is, $\ell \parallel n$. By uniqueness in **AP₂** one has $n = m$, and therefore $\lambda = \hat{n} = \hat{m}$.
- Now suppose $\alpha, \beta \in \ell_\infty$. Clearly ℓ_∞ is the only line in $\hat{\mathcal{L}}$ incident to both α and β.

PP₂. Exercise.

PP₃. Obvious from **AP₃** and the definition of the projectivisation. □

The converse operation of 'downgrading from projective to affine' involves choosing the line to be removed.

2.2.5. Definition (affinisation $\check{\mathbb{P}}_\lambda$ of a projective plane). Let $\mathbb{P} = (\mathcal{P}, \mathcal{L}, \in)$ be a projective plane. Fix one line $\lambda \in \mathcal{L}$. The *affinisation of* $\mathbb{P}$ *with respect to* λ is the incidence geometry $\check{\mathbb{P}}_\lambda = (\check{\mathcal{P}}, \check{\mathcal{L}}, \in)$ defined as follows:

- for each $\mu \in \mathcal{L} \setminus \{\lambda\}$, set $\check{\mu} = \mu \setminus \lambda$;
- $\check{\mathcal{P}} = \mathcal{P} \setminus \lambda$;
- $\check{\mathcal{L}} = \{\check{\mu} : \mu \in \mathcal{L} \setminus \{\lambda\}\}$.

Be careful that not all our notation reflects dependence on λ.

2.2.6. Remarks.

- The isomorphism type of $\check{\mathbb{P}}_\lambda$ depends on the line λ you chose to remove. Affinisation is not uniquely defined, one may not say '*the* affinisation' without specifying λ. (Cf. ''the projectivisation', which is well-defined.)
- In particular, *an* arbitrary affinisation of the projectivisation of an affine plane $\mathbb{A}$ may depend on the choice of line removed from $\hat{\mathbb{A}}$, and need not be isomorphic to $\mathbb{A}$.
- But starting from projective $\mathbb{P}$ and letting $\mathbb{A} = \check{\mathbb{P}}_\lambda$, one has $\hat{\mathbb{A}} \simeq \mathbb{P}$ *regardless of* λ.

See exercise 2.4.4.

2.3 Projectivising $\mathbb{A}^2(\mathbb{F})$, affinising $\mathbb{P}^2(\mathbb{F})$

Applying the projectivisation/affinisation procedures to $\mathbb{A}^2(\mathbb{F})$ and $\mathbb{P}^2(\mathbb{F})$ gives what one expects.

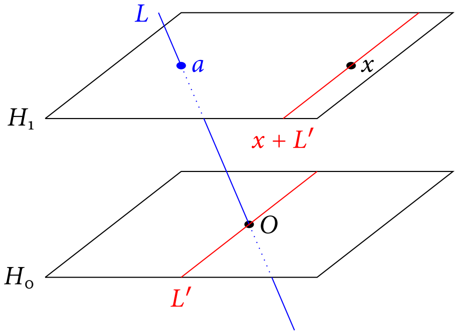
2.3.1. Proposition. *Let $\mathbb{F}$ be a skew-field. Then:*

(i) $\widehat{\mathbb{A}^2(\mathbb{F})} \simeq \mathbb{P}^2(\mathbb{F})$;

(ii) for any line, $\check{\mathbb{P}}^2(\mathbb{F}) \simeq \mathbb{A}^2(\mathbb{F})$.

Proof.

(i) We describe an isomorphism. Let $H_0 \leq \mathbb{F}^3$ be a vector plane and $x \in \mathbb{F}^3$ be a vector not in H_0. Let $H_1 = x + H_0$, an affine translate of H_0. Clearly $\mathbb{A}^2(\mathbb{F})$, H_0, and H_1 are isomorphic affine planes; in particular $\widehat{\mathbb{A}^2(\mathbb{F})}$ and $\widehat{H_1}$ are isomorphic projective planes. So it suffices to see that $\widehat{H_1}$ and $\mathbb{P}^2(\mathbb{F})$ are isomorphic, as follows.



$L \nleq H_0$ is mapped to a, while $L' \leq H_0$ is mapped to $[x + L']$.

Let $L \leq \mathbb{F}^3$ be a vector line. If $L \nleq H_0$, then L will intersect H_1 in a point say a; map L to a. If $L \leq H_0$, then there is no intersection, so L should be mapped to a point at infinity. In the previous construction, such points were the equivalence classes (directions) of lines of H_1; so map L to $[x + L]$, the direction of $x + L$ which is a line of the affine plane H_1. This maps vector lines in $\mathbb{F}^3$ to points in $\hat{H}_1$.

Now let $H \leq \mathbb{F}^3$ be a vector plane. If $H \neq H_0$ then $H \cap H_1$ is a line ℓ of the affine plane H_1; map H to $\hat{\ell}$ as in the projectivisation construction. If on the other hand $H = H_0$ then map H to ℓ_∞. This maps vector planes in $\mathbb{F}^3$ to lines in $\hat{H}_1$.

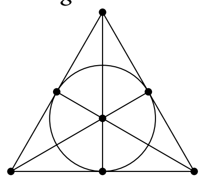
It is an exercise to finish the proof:

- check that we have a bijection between points of $\mathbb{P}^2(\mathbb{F})$ and points of $\widehat{H_1}$;
- check that we have a bijection between lines of $\mathbb{P}^2(\mathbb{F})$ and lines of $\widehat{H_1}$;
- check that these bijections preserve incidence.

(ii) Exercise. □

2.4 Exercises

2.4.1. Exercise. *Show that the following is the smallest projective plane, and identify it.*



The circle counts as a line.

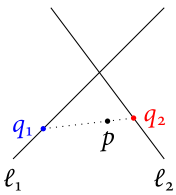
2.4.2. Exercise. *Let* $\mathbb{P}$ *be a projective plane with line* λ*. Show that* $\mathbb{A} = \check{\mathbb{P}}_\lambda$ *is an affine plane and* $\hat{\mathbb{A}} \simeq \mathbb{P}$*.*

2.4.3. Exercise. *Let* $\mathbb{F}$ *be a skew-field and* $\mathbb{P} = \mathbb{P}^2(\mathbb{F})$*. Prove that for any line one has* $\check{\mathbb{P}}_\lambda \simeq \mathbb{A}^2(\mathbb{F})$*.*

2.4.4. Exercise (the isomorphism type of $\check{\mathbb{P}}_\lambda$ depends on the line you remove)**.**

1. *Let* $\mathbb{A}$ *be an affine plane and* $\alpha \in \mathrm{Aut}(\mathbb{A})$ *be an automorphism. Prove that* α *extends to a unique automorphism of* $\hat{\mathbb{A}}$*.*
2. *Let* $\mathbb{P}$ *be a projective plane with two lines* λ_1, λ_2*. Let* $\mathbb{A}_i$ *be the affinisation with respect to* λ_i*. Suppose that* $\mathbb{A}_1 \simeq \mathbb{A}_2$*, and prove that there is* $\alpha \in \mathrm{Aut}(\mathbb{P})$ *sending* λ *to* μ*.*
3. *Deduce that the isomorphism type of* $\check{\mathbb{P}}_\lambda$ *may depend on the line you remove. Hint: there is a projective plane with whose automorphism group is not transitive on lines (see Exercise 3.4.6).*
4. *Also deduce that there exist two non-isomorphic affine planes with isomorphic projectivisations.*
5. *Prove however that* if $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$*, then* $\check{\mathbb{P}}_\lambda$ *does not depend on the line you remove.*

2.4.5. Exercise. *Let* $\mathbb{P}$ *be a projective plane. Let* $\ell_1 \neq \ell_2$ *be two lines and* $p \notin \ell_1 \cup \ell_2$*. Show that the following picture defines a bijection* $\ell_1 \simeq \ell_2$*.*



2.4.6. Exercise. *Let* $\mathbb{P}$ *be a* finite *projective plane.*

1. *For* ℓ *a line and* $p \notin \ell$*, give a bijection between* ℓ *and* $\mathcal{L}_p = \{m \in \mathcal{L} : p \in m\}$*.*
2. *Deduce that there is an integer* n *such that: • every line has* $n+1$ *points; • every points belongs to* $n+1$ *lines; • there are* n^2+n+1 *points and as many lines.*
3. *What is the value of* n *for* $\mathbb{P}^2(\mathbb{F}_q)$*, where* $\mathbb{F}_q$ *is the field of order* q*?*

Remark. The integer n is called the *order* of n. It has been conjectured that for finite $\mathbb{P}$, the order is always a prime power; this is open. It is not even known whether there is a projective plane of order 12; killing those of order 10 has been remarkably difficult. Finitary aspects are fascinating but of *astounding* complexity.[3]

2.4.7. Exercise. *Let $\mathbb{A}$ be a finite affine plane. Show that there is n such that: • every line has n points; • every point is on $n+1$ lines; • there are n^2 points in total; • there are n^2+n lines, falling in $n+1$ parallel classes of n lines each. Hint: this exercise belongs to § 2, not to § 1.*

2.4.8. Exercise. *Let $G = \mathrm{SO}_3(\mathbb{R})$. Let $I = \{i \in G : i^2 = 1 \neq i\}$ be the set of involutions. On I consider the relation $i \varepsilon j$ iff $(ij = ji \neq 1)$. Prove that $(I, I, \varepsilon) \simeq \mathbb{P}^2(\mathbb{R})$.*

3 The Desargues property

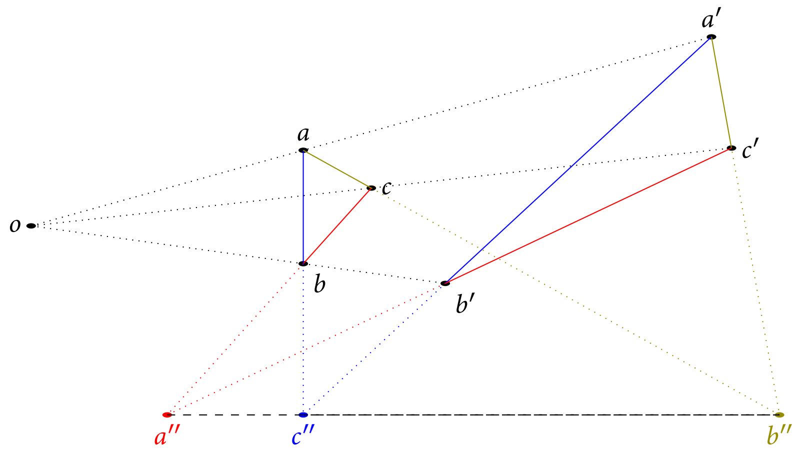
Abstract. The *Desargues property* is *not* 'yet another theorem' in geometry. It is a key combinatorial statement characterising those projective planes which are of the form $\mathbb{P}^2(\mathbb{F})$. § 3.1 describes the configuration and its affine versions. § 3.2 constructs projective planes which do *not* have the Desargues property. § 3.3 states the main result, *Hilbert's Coordinatisation Theorem*. (§§ 4 and 5 are devoted to proving it.)

3.1 Statement of the property

Recall that given two points $a \neq b$ of an affine or projective plane, (ab) denotes the unique line through them. It also equals (ba).

3.1.1. Definition (projective desarguesian plane)**.** A projective plane is *desarguesian* if it has the *projective Desargues property*, which is the following axiom.

> If abc, $a'b'c'$ are two triangles such that (aa'), (bb'), and (cc') meet, then $a'' = (bc) \cap (b'c')$, $b'' = (ac) \cap (a'c')$, and $c'' = (ab) \cap (a'b')$ are collinear.



One should practice drawing similar pictures, obtained as follows. 1. Choose a point o and draw three lines from it. 2. 'Hang' two triangles on them (each line contains exactly one vertex of each triangle). 3. Match sides of the triangles and draw intersections of matching sides. 4. The resulting intersections *are* collinear.

[3] C. Lam, The search for a finite projective plane of order 10, *Am. Math. Monthly*, 98 no. 4, pp. 305–318, April 1991.

3.1.2. Remark. No assumptions nor conclusions about o being on (a'')–(b'')–(c'').

The following definition is not for learning. One should be aware that depending on the position of the line at infinity, the projective Desargues property will have several, different-looking affine avatars.

3.1.3. Definition (affine desarguesian plane)**.** An affine plane is *desarguesian* if it has the following *affine Desargues property*, which is the following axiom.

> Let abc and $a'b'c'$ be triangles (involving six distinct points). Suppose that lines $(aa'), (bb'), (cc')$ are pairwise distinct, and either that they are parallel or that they meet at the same point. Then:
>
> - either $(ab) /\!/ (a'b')$, $(ac) /\!/ (a'c')$, and $(bc) /\!/ (b'c')$;
> - or $(ab) /\!/ (a'b')$ but $(ac) \cap (a'c') = \{b''\}$ and $(bc) \cap (b'c') = \{a''\}$ satisfy $(a''b'') /\!/ (ab)$;
> - or two similar cases (involving the intersection c'' of (ab) and $(a'b')$);
> - or a'', b'', c'' are collinear.

The following is straightforward.

3.1.4. Lemma.

(i) Let $\mathbb{A}$ be an affine plane. Then $\mathbb{A}$ is desarguesian iff $\hat{\mathbb{A}}$ is.

(ii) Let $\mathbb{P}$ be a projective plane with fixed line λ. Then $\mathbb{P}$ is desarguesian iff $\check{\mathbb{P}}_\lambda$ is.

We therefore focus on the projective version.

3.1.5. Remark. There is a form of 'converse'. Let Desargues' be the following property (which a plane may have or not).

> If two triangles abc and $a'b'c'$ are such that the intersection points $a'' = (bc) \cap (b'c')$, $b'' = (ac) \cap (a'c')$ and $c'' = (ab) \cap (a'b')$ are collinear, then lines (aa'), (bb') and (cc') are concurrent.

It turns out that this condition is *equivalent* to the original Desargues property. This is proved in § 12.

Some projective planes satisfy the Desargues property ($\mathbb{P}^2(\mathbb{F})$ does, which will be proved); some do not (§ 3.2). Hilbert characterised which do.

3.2 Free planes and non-desarguesian planes

We shall construct non-desarguesian planes by use of a *free construction*. The general idea behind such constructions is to add everything one needs, and nothing more. Here it will mean adding lines to connect any two points, but also adding points to intersect any two lines. Of course this creates new lines and points, so it must be done inductively.

3.2.1. Definition (free plane[4])**.** Let $\Gamma_0 = (\mathcal{P}_0, \mathcal{L}_0, \in)$ be an incidence geometry.

- Construct a sequence of geometries $\Gamma_n = (\mathcal{P}_n, \mathcal{L}_n)$ by the following induction. Let n be given.

[4]Free projective planes where introduced in: M. Hall. Projective Planes. Trans. Amer. Math. Soc. 54 (1943), 229-277.

- Let X_n be the set of unordered pairs $\{a, b\}$ of distinct points of $\mathcal{P}_n$ which are not connected by a line of $\mathcal{L}_n$. Also let Y_n be the set of unordered pairs $\{\ell, m\}$ of distinct lines of $\mathcal{L}_n$ which do not meet in a point of $\mathcal{P}_n$.
- For each $\{a, b\} \in X_n$, add a new line $\ell_{\{a,b\}}$ incident to a and b, but to no other point. Also for each $\{\ell, m\} \in Y_n$, add a new point $p_{\{\ell,m\}}$ on both ℓ and m, but on no other line.
- Let $\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{\ell_{\{a,b\}} : \{a, b\} \in X_n\}$ and $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \{p_{\{\ell,m\}} : \{\ell, m\} \in Y_n\}$.
- Finally let $\mathcal{P} = \bigcup_{\mathbb{N}} \mathcal{P}_n$ and $\mathcal{L} = \bigcup_{\mathbb{N}} \mathcal{L}_n$.

Then $\Gamma = (\mathcal{P}, \mathcal{L})$ is called the *plane freely generated by* Γ_0.

3.2.2. Remarks.

- In some sources one adds lines at odd stages and points at even stages, but this is less symmetric (§ 12).
- Starting with no points and no lines, or just collinear points, or concurrent lines, nothing will happen.
- Starting with Γ_0 having bad incidence properties, the free geometry will retain them.

3.2.3. Definition (admissible incidence geometry). An incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ is *admissible* if there are no two distinct lines sharing two distinct points.

The forbidden configuration.

3.2.4. Proposition. *Let* $\Gamma_0 = (\mathcal{P}_0, \mathcal{L}_0, \in)$ *be an admissible incidence geometry satisfying* **PP$_3$**. *Then the plane freely generated by* Γ_0 *is a projective plane.*

Proof. As usual, three things must be checked.

PP$_1$. Let $a \neq b \in \mathcal{P}$ be two distinct points. Say that $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$; we may assume $n \geq m$.

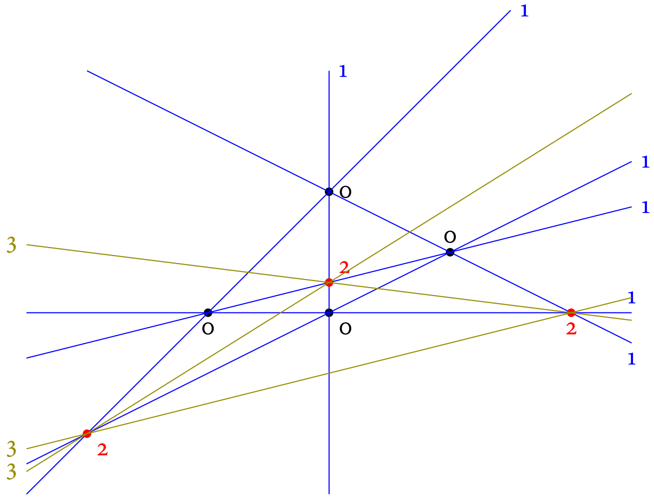
First suppose that $n = 0$ and a, b are already connected by a line $\ell \in \mathcal{L}_0$. Then ℓ is unique as such; moreover, we never introduce another line through a and b since $\{a, b\} \notin X_n$. So ℓ remains unique even in $\mathcal{L}$.

Now suppose that there is no $\ell \in \mathcal{L}_0$ containing a and b. Regardless of the value of n, at stage $n + 1$ we have introduced a line ℓ joining a and b. The argument for uniqueness of ℓ is similar.

PP$_2$. Entirely similar.

PP$_3$. This is almost by assumption but not entirely. There are four points $a, b, c, d \in \mathcal{P}_0 \subseteq \mathcal{P}$ which are not collinear in $\mathcal{L}_0$. We must see that they remain non-collinear in the sense of $\mathcal{L}$. But at stage 1, when we add line $\ell = (ab)$, it does not contain c. And we never add another line through a and b. So there never is a line containing a, b, c: they remain non-collinear also in $\mathcal{L}$. Likewise for the other triples. □

3.2.5. Example. Start with the admissible incidence geometry consisting of four points and no lines.



Numbers indicate the stages of appearance of points/lines. Imagine the following next steps.

This example is vacuously desarguesian: there are *no* Desargues configurations.

3.2.6. Remark. More generally, let Γ^κ be the free plane generated by κ points and no lines (κ may be infinite). Their class behaves roughly like the class of free groups. Here are a few (non-trivial) properties.

- A subplane of a Γ^κ is another Γ^λ (λ need not be smaller than κ).
- As κ varies the Γ^κ are pairwise non-isomorphic.
- However, they all have the same elementary theory.

The model theory of the free projective plane has been studied only recently.[5]

3.2.7. Theorem. *There exist non-desarguesian projective and affine planes.*

Proof. Start with the admissible incidence geometry Γ_0 consisting of:

- ten points $\mathcal{P}_0 = \{o, a, b, c, a', b', c', a'', b'', c''\}$,
- $\mathcal{L}_0$ = all lines involved in the Desargues configuration *except line* $(a''b''c'')$.

Let $\mathbb{P}$ be the plane freely generated by Γ_0. By Proposition 3.2.4, $\mathbb{P}$ is a projective plane. In Γ_0 there is no line joining a'', b'', c'', and in the construction we never add a line simultaneously joining three existing points. So in $\mathbb{P}$ there is no line joining a'', b'', c''. Therefore $\mathbb{P}$ does not satisfy the Desargues property. □

3.2.8. Remarks.

- Although the example of a non-desarguesian plane we gave is obviously infinite, *there exist* finite*, non-desarguesian projective planes.*
- Hilbert's *Grundlagen der Geometrie*, Theorem 33, contains a completely different example of a non-desarguesian plane. His construction is rather involved.

[5] T. Hyttinen and G. Paolini. First-order model theory of free projective planes. Ann. Pure Appl. Logic 172 (2) (2021), paper no. 102888.

3.3 Hilbert's Coordinatisation Theorem

3.3.1. Theorem (Hilbert coordinatisation). *Let $\mathbb{P}$ be a projective plane. Then the following are equivalent:*

(i) *$\mathbb{P}$ is desarguesian;*

(ii) *there is a skew-field (unique up to isomorphism) $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$;*

(iii) *$\mathbb{P}$ is isomorphic to a plane in some projective, 3-dimensional space (Definition 4.1.1 below).*

3.3.2. Remark. The Desargues property is therefore a strong dividing line in the theory of projective planes; the latter come in two sorts:

- desarguesian projective planes (of the form $\mathbb{P}^2(\mathbb{F})$ for some skew-field): our knowledge is satisfactory, viz. as satisfactory as our understanding of skew-fields;
- non-desarguesian projective planes (*not* of the form $\mathbb{P}^2(\mathbb{F})$ for any skew-field): we know extremely little, and presumably there is little to say in general.

In particular, contemporary computers are not powerful enough to help us classify (?) finite projective planes. One would need new ideas.

Notice that the free, non-desarguesian construction of § 3.2 shows that there exist non-coordinatisable projective planes.

3.4 Exercises

3.4.1. Exercise. *Return to § 3.1; we say that $(o; a, b, c, ; a', b', c'; a'', b'', c'')$ is a* Desargues configuration *seen from o.*

1. *Check that $(a''; b, c'', b'; c, b'', b'; a', o, a)$ is a Desargues configuration seen from a''.*
2. *From how many points is it a Desargues configuration?*
3. *Suppose $\mathbb{P}$ has the Desargues property. Prove that it has the 'converse Desargues property' (hint: a''). (Such tricks are explained in § 12.)*

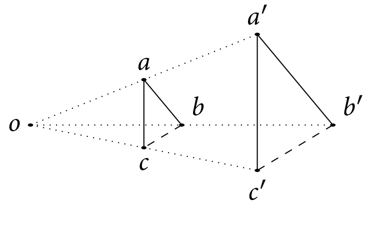
3.4.2. Exercise (harder). *The 'generic' Desargues configuration is the one where $o \notin (a''b'')$: there are 10 points, 10 lines, and every point is on exactly 3 lines.*

Determine the automorphism group of the generic Desargues configuration, viz. the group of bijections of the 10 points and 10 lines which preserve all incidence relations.

3.4.3. Exercise. *Prove the Desargues property in $\mathbb{P}^2(\mathbb{F})$ using projective coordinates. (If any simpler, prove all affine versions using affine coordinates, then conclude.)*

3.4.4. Exercise (affine versions).

1. *Removing various lines, draw all affine versions of the Desargues property. Below is one.*



2. *Prove that a projective plane $\mathbb{P}$ is desarguesian iff for any line λ, its affinisation $\check{\mathbb{P}}_\lambda$ is.*

3. *Prove that an affine plane $\mathbb{A}$ is desarguesian iff its projectivisation is.*

3.4.5. Exercise. *Let Γ_0 be an admissible incidence geometry satisfying* **PP$_3$**. *Show that if the plane freely generated by Γ_0 is some Γ_n from the construction, then Γ_0 was a projective plane already. Hint: exercise 2.4.5.*

3.4.6. Exercise.

1. *Let Γ_0 be four points with no lines, and Γ be the free projective plane generated by Γ_0. Prove that no substructure of Γ is isomorphic to the* Fano plane *of Exercise 2.4.1. Hint: consider the least n such that Γ_n contains a copy of the Fano plane.*

2. *Let Δ_0 be the Fano plane with an extra (not connected) point. Let Δ be the free projectiev plane generated by Δ_0. Show that Δ contains a unique copy of the Fano plane.*

3. *Deduce that there exist projective planes whose group of automorphisms does not permute lines transitively.*

3.4.7. Exercise (The Moulton plane[6])**.** *Equip $\mathbb{R}^2$ with the following lines:*

- *all ordinary lines with non-positive slope (including horizontal and vertical lines);*
- *for each $a > 0$ and $x_0 \in \mathbb{R}^2$, the set $\ell_{a,x_0} = \{(x, 2a(x-x_0)) : x \in \mathbb{R}_{\leq 0}\} \cup \{(x, a(x-x_0)) : x \in \mathbb{R}_{\geq 0}\}$.*

1. *Draw a picture.*

2. *Check that this is an* affine *plane.*

3. *Prove that at least one instance of the affine Desargues properties fails. Hint: take an ordinary Desargues configuration, translate and rotate so that nine points out of ten are above zero, and two lines out of three passing through this point are 'unbent'.*

4. *Conclude that there is a non-desarguesian* projective *plane.*

4 A detour through space

Abstract. § 4.1 introduces *projective* 3*-spaces*. § 4.2 (optional) proves a non-planar version of the Desargues property in such spaces. § 4.3 proves part of Hilbert's Coordinatisation Theorem. (The argument continues in § 5.)

We want to prove the following.

Theorem (Hilbert coordinatisation)**.** *Let $\mathbb{P}$ be a projective plane. Then the following are equivalent:*

(i) $\mathbb{P}$ *is desarguesian;*

(ii) there is a skew-field (unique up to isomorphism) $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$;

(iii) $\mathbb{P}$ is isomorphic to a plane in some projective, 3-dimensional space.

We first explain the missing terminology.

[6] F. Moulton, Simple non-desarguesian geometry, *Trans. Amer. Math. Soc.*, April 1902

4.1 Projective 3-spaces

4.1.1. Definition (projective 3-space)**.** A *projective 3-dimensional space* is an incidence structure $(\mathcal{P}, \mathcal{L}, \Pi)$ consisting of points, lines, and planes (and incidence relations), satisfying the following axioms $\mathbf{PS_1}$–$\mathbf{PS_6}$.

$\mathbf{PS_1}$**.** Any two points are uniquely collinear.

$\mathbf{PS_2}$**.** Any three non-collinear points are uniquely coplanar.

$\mathbf{PS_3}$**.** Any line and plane meet.

$\mathbf{PS_4}$**.** Any two planes share (at least) a line.

$\mathbf{PS_5}$**.** There are four non-coplanar points no three of which are collinear.

$\mathbf{PS_6}$**.** Every line has at least three points.

4.1.2. Example. Let $\mathbb{F}$ be a skew-field. Then $\mathbb{P}^3(\mathbb{F})$, where points are (left-)vector lines in $\mathbb{F}^4$, lines are 2-dimensional (left-)vector subspaces of $\mathbb{F}^4$, and planes are (left-)vector hyperplanes of $\mathbb{F}^4$, is a projective space (for set-theoretic incidence).

It will be a consequence of Hilbert Coordinatisation that *every projective 3-space is of this form.* (Cf. dimension 2: there exist projective planes not of the form $\mathbb{P}^2(\mathbb{F})$.)

Like in Proposition 1.1.4, every projective 3-space is isomorphic to one where incidence relations are of the form $p \in \ell \subseteq \pi$. This simplifies notation and terminology.

4.1.3. Lemma. *Let $\mathbb{S} = (\mathcal{P}, \mathcal{L}, \Pi)$ be a projective 3-space, and let $\pi \in \Pi$. Let $\mathcal{P}_\pi = \{p \in \mathcal{P} : p \in \pi\}$ and $\mathcal{L}_\pi = \{\ell \in \mathcal{L} : \ell \in \pi\}$. Then $(\mathcal{P}_\pi, \mathcal{L}_\pi, \in)$ is a projective plane.*

Proof. See exercise 4.4.2. □

Therefore 'isomorphic to a plane in some projective, 3-dimensional space' means 'isomorphic to $\mathcal{P}_\pi$ for some π in some projective, 3-dimensional space'.

Proof of (ii)⇒(iii): $\mathbb{P}^2(\mathbb{F})$ embeds into a projective 3-space. We prove that for any skew-field, $\mathbb{P}^2(\mathbb{F})$ embeds into $\mathbb{P}^3(\mathbb{F})$.

This is trivial in projective/homogeneous coordinates; one not at ease with them would argue as follows. Using *linear* coordinates, embed $\mathbb{F}^3$ into $\mathbb{F}^4$, say on the 'horizontal' hyperplane: map (x, y, z) to $(x, y, z, 0)$. This is a linear embedding, so it certainly preserves collinearity. Moreover, it preserves inclusion of vector subspaces. So it induces well-defined, injective maps :

$$\begin{aligned} f&: \{\text{vector lines of } \mathbb{F}^3\} \to \{\text{vector lines of } \mathbb{F}^4\}, \\ g&: \{\text{vector planes of } \mathbb{F}^3\} \to \{\text{vector planes of } \mathbb{F}^4\}, \end{aligned}$$

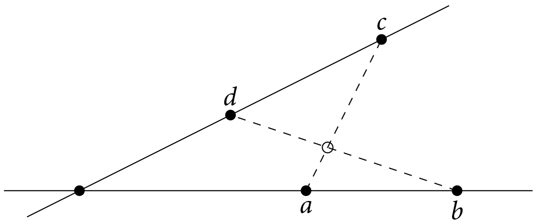
which preserve $\leq$. These define an embedding of $\mathbb{P}^2(\mathbb{K})$ into $\mathbb{P}^3(\mathbb{K})$. □

4.1.4. Remark (higher-dimensional projective geometries)**.** One may wish to capture n-dimensional projective spaces.

- When n is given, one can always introduce n distinct types of objects (points, lines, and so on) and give a long list of axioms. This is clumsy.

- One can consider only points $\mathcal{P}$ and 'subspaces' $\mathcal{S}$ equipped with a dimension function $\mathcal{S} \to \mathbb{N}$. But this is unsatisfactory, logically speaking, as the description of the geometry now relies on the purely non-geometric notion of the integers.
- An elegant solution, 'à la Veblen', is the following. Call *projective space* any incidence geometry satisfying $\mathbf{PP_1}$, $\mathbf{PP_3}$, and the *Pasch axiom*:

  if a, b, c, d are points such that $(ab) \cap (cd) \neq \varnothing$, then $(ac) \cap (bd) \neq \varnothing$.



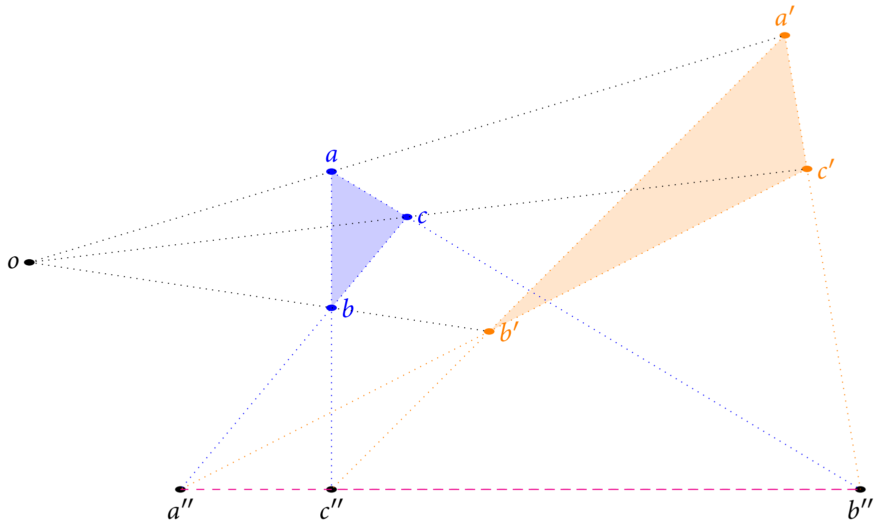
The *Pasch axiom* describes coplanarity without mentioning planes.

4.2 Non-planar Desargues configurations

This subsection is not necessary but helps gain some intuition of the next.

4.2.1. Lemma. *Work inside a projective 3-space $\mathbb{S}$. Suppose $(o; a, b, c; a', b'; a'', b'', c'')$ is a* non-planar *Desargues configuration in $\mathbb{S}$. Then a'', b'', c'' are collinear.*

Proof. By assumption, abc and $a'b'c'$ do not define the same plane: otherwise *all* points and lines would be in that plane. Say (abc) is in some plane π_1 and $(a'b'c')$ in some other plane $\pi_2 \neq \pi_1$.



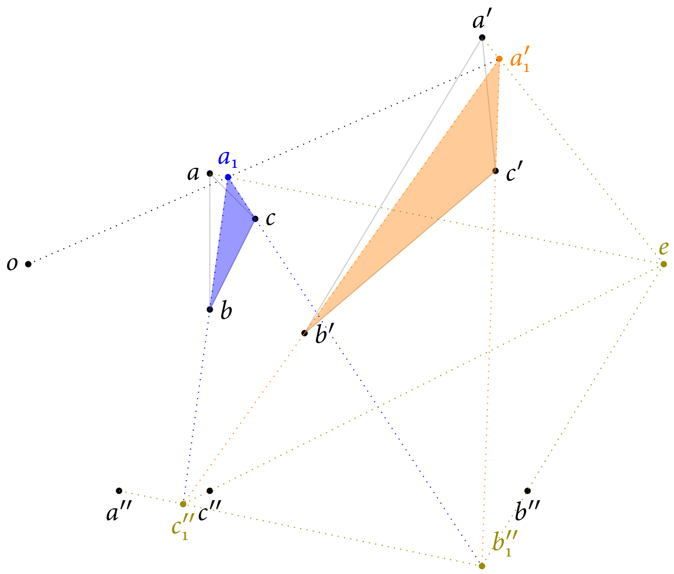
Then observe how $a'' \in (bc) \subseteq \pi_1$; likewise, $b'', c'' \in \pi_1$. But similarly, $a'' \in (b'c') \subseteq \pi_2$; likewise, $b'', c'' \in \pi_2$. So $a'', b'', c'' \in \pi_1 \cap \pi_2$. One may prove that the intersection of two distinct planes is *exactly* a line (exercise 4.4.2): whence collinearity of the three points. □

4.3 'Embeddable implies desarguesian'

We prove that if a projective plane is isomorphic to a plane in a projective 3-space, then it is desarguesian.

Proof of (iii)⇒(i). Let $\mathbb{P}$ be a projective plane isomorphic to some plane of a projective 3-space $\mathbb{S}$.

Consider a Desargues configuration $(o; a, b, c; a', b', c; a'', b'', c'')$ taking place in $\mathbb{P}$. We want to show collinearity of a'', b'', c''. The idea is to slightly 'lift' the picture, work in $\mathbb{S}$, and then project back.



Imagine 'slightly rotating' the planar triangles by 'slightly lifting' the vertices a and a' into a_1 and a'_1, still preserving collinearity of o, a_1, a'_1. Mathematically this is achieved through choosing $e \notin \pi$ then $a1 \in (ae)$.

- Let $e \in \mathbb{S} \setminus \mathbb{P}$.
- Let $a_1 \in (ae)$ be neither a nor e; there is such a point by $\mathbf{PS_6}$.
- Also let $a'_1 = (oa_1) \cap (ea')$. Having changed triangles, we must consider new intersections.
- Finally let $b''_1 = (a_1c) \cap (a'_1c')$ and $c''_1 = (a_1b) \cap (a'_1b')$; notice that a'' however remains unchanged.

Step 1. Points b''_1 and c''_1 are well-defined ; triangles a_1bc and $a'_1b'c'$ satisfy the assumption of the Desargues property and are not coplanar.

Verification. First, the 'Desargues intersections' a'', b''_1, c''_1 do exist; this is no longer trivial as lines in $\mathbb{S}$ need not intersect. Point $a'' = (bc) \cap (b'c')$ has not changed. By $\mathbf{PS_2}$, let π' be the plane containing o, a_1, c. It then contains $a'_1 \in (oa_1)$ and $c' \in (oc)$, so π' also contains lines $(a_1 c)$ and $(a_1' c')$, which therefore meet: intersection b_1'' is well-defined. Similar argument for c_1''.

By construction, o, a_1, a_1' are collinear. Since other vertices have not changed, triangles $a_1 bc$ and $a'1b'c'$ meet the Desargues assumption. Moreover, these triangles are not in the same plane. Otherwise that plane would be $\mathbb{P}$; since $a, a_1 \in \mathbb{P}$, one then has $e \in (aa_1) \subseteq \mathbb{P}$, a contradiction. ◇

Step 2. Points a'', b_1'', c_1'' are collinear.

Verification. This is the argument given in Lemma 4.2.1. Using axiom **PS₂**, let π_1 be the plane containing a_1, b, c and π_2 be the plane containing a_1', b', c'. Clearly these planes are distinct, so $\pi_1 \cap \pi_2$ contains a line ℓ.

By Lemma 4.1.3, $\pi_1 \cap \pi_2 = \ell$ is a line. Observe how: $a'' \in (bc) \cap (b'c') \subseteq \pi_1 \cap \pi_2 = \ell$, and likewise for b'' and c''. Finally a'', b'', c'' are on ℓ, hence collinear. ◇

We now project back. Let p be the *projection* function from e onto π, defined as follows. For any point $x \neq e$, let $p(x)$ be the meeting point of line (ex) and plane π, which does not contain it. (The value of p at e remains undefined.) Notice that:

- p is the identity on π;
- p preserves incidence, i.e. if $x \in \ell$, then $p(x) \in p(\ell)$.

Step 3. $p(b_1'') = b''$ and $p(c_1'') = c''$.

Verification. Since $(ea_1) = (ea)$ and $a \in \pi$, one has $p(a_1) = a$. But also $(ea_1') = (ea')$ so $p(a_1') = a'$. In particular,

$$\begin{aligned}\{p(b_1'')\} &= p\left((a_1 c) \cap (a_1' c')\right) \\ &= p((a_1 c)) \cap p((a_1' c')) \\ &= (p(a_1)p(c)) \cap (p(a_1')p(c')) \\ &= (ac) \cap (a'c') \\ &= \{b''\}.\end{aligned}$$

One obtains $p(c_1'') = c''$ likewise. ◇

Since p preserves incidence (and therefore collinearity), the images of a'', b_1'', c_1'' remain collinear. Therefore $p(a'') = a'', p(b_1'') = b''$, and $p(c_1'') = c''$ are collinear. □

4.3.1. Remarks.

- Essentially the ideas here come from Desargues (who treated only $\mathbb{F} = \mathbb{R}$).
- Baldwin and Howard gave a direct proof (i)⇒(iii) without introducing coordinates.[7] It is quite involved.

It remains to prove that a desarguesian projective plane can be coordinatised. This was first done by Hilbert and we sketch his method in § 5.

[7] J. Baldwin. Formalization, primitive concepts, and purity. Rev. Symb. Log. 6 (1), 87–128. 2013.

4.4 Exercises

4.4.1. Exercise. *Return to Remark 4.1.4. Prove that Pasch's axiom is equivalent to:*

> *if a line meets two sides of a triangle, it also meets the third side.*

4.4.2. Exercise. *Let* $\mathbb{S} = (\mathcal{P}, \mathcal{L}, \Pi)$ *be a projective 3-space. Prove the following.*

1. *If* $\pi_1 \neq \pi_2$ *are distinct planes sharing line* ℓ*, then* $\pi_1 \cap \pi_2 = \ell$.
2. *If* p, q *are points then there are at least two planes containing them.*
3. *If* $p \neq q$ *are distinct points on plane* π*, then* $(pq) \subseteq \pi$.
4. *Lemma 4.1.3.*

5 The coordinatisation theorem: Hilbert's version

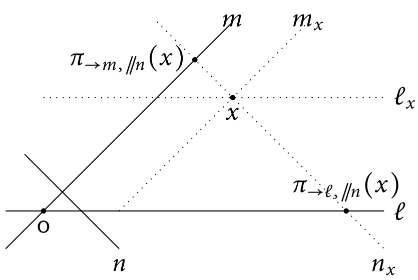
Abstract. A sketch of Hilbert's proof that a desarguesian projective plane can be coordinatised. The argument is long and pictures are increasingly involved, although the conceptual level remains basic. We just give the first steps; some further steps are left as exercises; some should just be read from Hilbert's book, or imagined. In more advanced sections (§§ 10–11), we shall give abstract arguments using group theory.

Let $\mathbb{P}$ be a desarguesian projective plane. Here is the global strategy.

1. Fix some line $\lambda \in \mathcal{L}(\mathbb{P})$. Let $\mathbb{A} = \check{\mathbb{P}}_\lambda$; then $\mathbb{A}$ is a desarguesian *affine* plane. *Coordinatisation actually takes place in* $\mathbb{A}$.
2. Fix one line ℓ of $\mathbb{A}$ and cleverly define two operations $+, \cdot$ on it. Then $\mathbb{F} = (\ell; +, \cdot)$ is a skew-field. (Checking it takes time. But arguments are repetitive and one quickly gets the general ideal.)
3. Coordinatise $\mathbb{A}$ using $\mathbb{F}$, so $\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$. (This is actually not as obvious as it seems, and Desargues is used again.)
4. Finally conclude $\mathbb{P} \simeq \hat{\mathbb{A}} \simeq \widehat{\mathbb{A}^2(\mathbb{F})} \simeq \mathbb{P}^2(\mathbb{F})$.

We do not cover (2) entirely, and we entirely omit (3). Details are to be found in Hilbert's *Grundlagen der Geometrie*, chapter V. The proof starts here.

5.1 The frame



Notation for directions. Let ℓ, m, n be three pairwise non-parallel lines (later we shall require them to be non-concurrent).

For $x \in \mathbb{A}$ we let:

- ℓ_x be the line parallel to ℓ through x;
- m_x be the line parallel to m through x;
- n_x be the line parallel to n through x.

Notation for projections. Also construct:

$$\begin{array}{rccc} \pi_{\to \ell, /\!/ n}: & \mathbb{A} & \to & \ell \\ & x & \mapsto & \ell \cap n_x. \end{array}$$

Since $n_x /\!/ n$ is not parallel to ℓ, this makes sense. Define $\pi_{\to m, /\!/ n} : \mathbb{A} \to m$ similarly. Notice that $\pi_{\to \ell, /\!/ n}$ is the identity on ℓ (and similarly for $\pi_{\to m, /\!/ n}$).

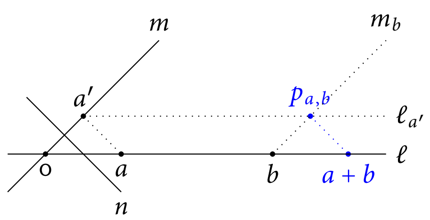
The prime function. Now consider:

$$\begin{array}{ccc} \ell & \to & m \\ a & \mapsto & a' := n_a \cap m \end{array}$$

clearly a bijection between ℓ and m. For any $x \in \mathbb{A}$ one has $(\pi_{\to \ell, /\!/ n}(x))' = \pi_{\to m, /\!/ n}(x)$.

Finally let $o = \ell \cap m$ and notice $o' = o$.

5.2 Addition



5.2.1. Notation. For a, b in ℓ, define:

- $p_{a,b} = \ell_{a'} \cap m_b$ (p is for 'plus');
- $a + b = \pi_{\to \ell, /\!/ n}(p_{a,b})$.

5.2.2. Proposition. *$(\ell; +)$ is an abelian group with identity* o.

Proof. The proof is a series of lemmas.

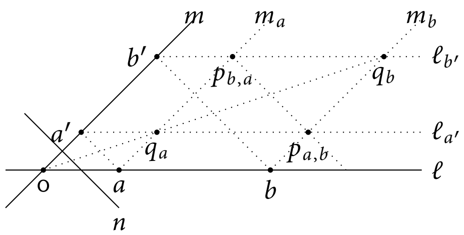
5.2.3. Lemma. *$+$ is well-defined; for any $a \in \ell$ one has $o + a = a + o = a$.*

Proof. Since ℓ and m are not parallel and $/\!/$ is an equivalence relation, $p_{a,b}$ is uniquely defined. So is $a + b \in \ell$. Neutrality is obvious as the construction then degenerates: $\ell_{o'} = \ell$ so $p_{o,a} = \ell \cap m_a = a$, and $m_o = m$ so $p_{a,o} = \ell_{a'} \cap m = a'$, whence

$a + \mathrm{o} = \pi_{\to \ell, /\!/ n}(a') = a.$ □

5.2.4. Lemma. *Addition is commutative.*

Proof. We may suppose that neither a nor b is o. We may also suppose $a \neq b$. Consider the following picture.

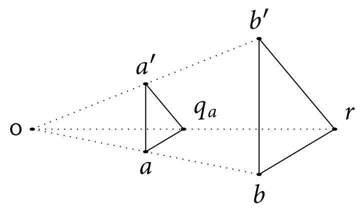


Introduce:

- $q_a = m_a \cap \ell_{a'}$;
- $q_b = m_b \cap \ell_{b'}$.

Step 1. Points o, q_a, q_b are collinear.

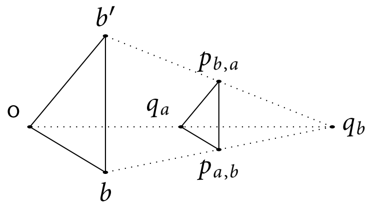
Verification. Let $r = (\mathrm{o}q_a) \cap \ell_{b'}$. Consider the configuration $(\mathrm{o}; a', q_a, a; b', r, b)$.



Then $(a'a) = n_a$ and $(b'b) = n_b$ are parallel; also, $(a'q_a) = \ell_{a'}$ and $(b'r) = \ell_{b'}$ are parallel. So by the Desargues property, $(aq_a) = m_a$ and (br) are parallel. Therefore also $(br) /\!/ m_a /\!/ m_b$; now the only parallel to m_b through b being m_b, one has $r \in (br) = m_b$. Since also $r \in \ell_{b'}$ by construction, it must be $r = q_b$. By construction, $\mathrm{o}, q_a, q_b = r$ are now collinear. ◇

Step 2. Lines $(p_{b,a}\, p_{a,b})$ and n are parallel.

Verification. Consider the Desargues configuration $(q_b; p_{b,a}, q_a, p_{a,b}; b', o, b)$.



Then $(b'o) = m$ and $(p_{b,a}q_a) = m_a$ are parallel, and also $(ob) = \ell$ and $(q_a p_{a,b}) = \ell_{a'}$ are parallel. By the Desargues property, $(b'b) = n_b$ and $(p_{b,a}p_{a,b})$ are parallel. ◇

Since $(p_{b,a}p_{a,b}) \parallel n$, one has $n_{p_{b,a}} = n_{p_{a,b}}$. Hence $b + a = \pi_{\to\ell,\parallel n}(p_{b,a}) = \pi_{\to\ell,\parallel n}(p_{a,b}) = a + b$. □

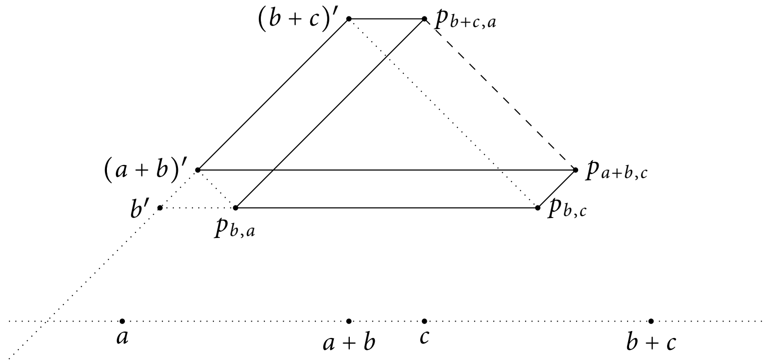
5.2.5. Lemma. *+ has opposites.*

Proof. Let $a \in \ell$; we may assume that $a \neq 0$. We must find $b \in \ell$ such that $a + b = 0$; by commutativity this will suffice. Let $q = n_o \cap \ell_{a'}$ (which makes sense as n and ℓ are not parallel), then $b = \ell_o \cap m_q$.

We claim that $a + b = 0$. Indeed, $p_{a,b} = \ell_{a'} \cap m_b = \ell_{a'} \cap m_q = q \in n_o$, so $a + b = \pi_{\to\ell,\parallel n}(p_{a,b}) = \pi_{\to\ell,\parallel n}(q) = 0$. □

5.2.6. Lemma. *+ is associative.*

Proof. Let $a, b, c \in \ell$; we may suppose that all three are distinct, and distinct from 0. Consider the following picture.



- We know that $(p_{b,a}\ p_{b,c}) = \ell_{b'}$ is parallel to $((a+b)'\ p_{(a+b),c)}) = \ell_{(a+b)'}$, which is parallel to $((b+c)'\ p_{b+c,a}) = \ell_{b+c}$.

- We also know that $((a+b)'\,(b+c)') = m$ is parallel to $(p_{b,a}\,p_{b+c,a}) = m_a$, which is parallel to $(p_{b,c}\,p_{a+b,c}) = m_c$.
- Finally we know that $a+b = b+a$, so $p_{b,a} \in n_{(b+a)} = n_{a+b}$. This means that $((a+b)'\,p_{b,a}) = n_{a+b}$ is parallel to $n_{b+c} = ((b+c)\,(b+c)') = (p_{b,c}\,(b+c)')$.

By exercise 5.5.1, $(p_{b+c,a}\,p_{a+b,c})$ is parallel to n. This means:

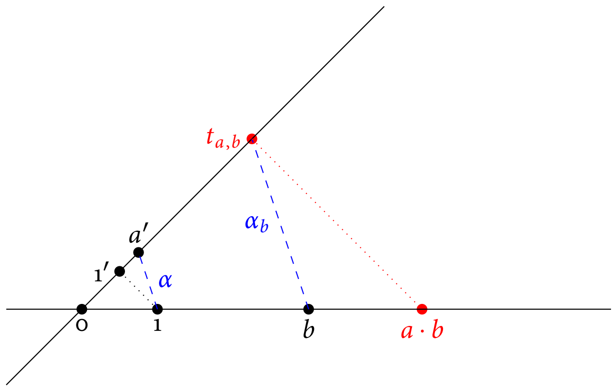
$$(b+c)+a = \pi_{\to\ell,/\!/n}(p_{b+c,a}) = \pi_{\to\ell,/\!/n}(p_{a+b,c}) = (a+b)+c.$$

Consequently $a+(b+c) = (a+b)+c$, and addition is associative. □

This completes the study of addition. □

5.3 Multiplication

Let $1 = \ell \cap n$; since ℓ, m, n not to concur, one has $1 \neq 0$.



5.3.1. Notation. For $a, b \in \ell$, define:

- $\alpha = (1a')$;
- $t_{a,b} = \alpha_b \cap m$ (t is for 'times');
- $a \cdot b = \pi_{\to\ell,/\!/n}(t_{a,b})$.

5.3.2. Proposition. *$(\ell \setminus \{0\};\cdot)$ is a group with identity* 1.

Proof. Exercise 5.5.2. □

5.4 Skew-field and coordinatisation

5.4.1. Proposition. *$(\ell;+,\cdot)$ is a skew-field.*

Both $+$ and $\cdot$ are well-understood separately, so only distributive laws remain to be checked; since $\cdot$ need not be commutative, there are two equations to check, namely $a\cdot(b+c) = a\cdot c + b\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$. This is extraordinarily painful, though always by repeated use of the Desargues property. One should have a (brief) look at Hilbert's *Grundlagen der Geometrie*.

5.4.2. Proposition. $\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$.

Proof. Not as trivial as it seems. One key step is to show that the isomorphism type of $\mathbb{F} = (\ell; +, \cdot)$ depends on *none* of the choices we made (we fixed $\ell, m, n, 0, 1, \dots$). This uses the Desargues property. Then 'the same coordinates can be used everywhere', and one retrieves $\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$. □

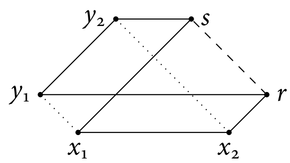
5.4.3. Corollary. *$\mathbb{F}$ is unique up to isomorphism.*

Proof. Suppose $\mathbb{P}$ is desarguesian and gives rise to two fields $\mathbb{F}$ and $\mathbb{F}'$. Then $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$. But when applied to $\mathbb{P}^2(\mathbb{F})$, one can check step by step that Hilbert's method yields $\mathbb{F}$. It follows that $\mathbb{F}' \simeq \mathbb{F}$: the coordinatising skew-field is unique up to isomorphism. □

5.4.4. Remark (coordinatising the uncoordinatisable)**.** One can still attempt to coordinatise non-desarguesian affine planes using *weaker* algebraic structures than skew-fields. So-called *ternary rings* serve this purpose. A ternary ring is a pair (R, T) where R is a set and $T\colon R^3 \to R$ a *ternary* operation of some form, which aims at capturing the behaviour of $(a, x, b) \mapsto ax + b$. These algebraic structures are arguably not 'core mathematics'. [8]

5.5 Exercises

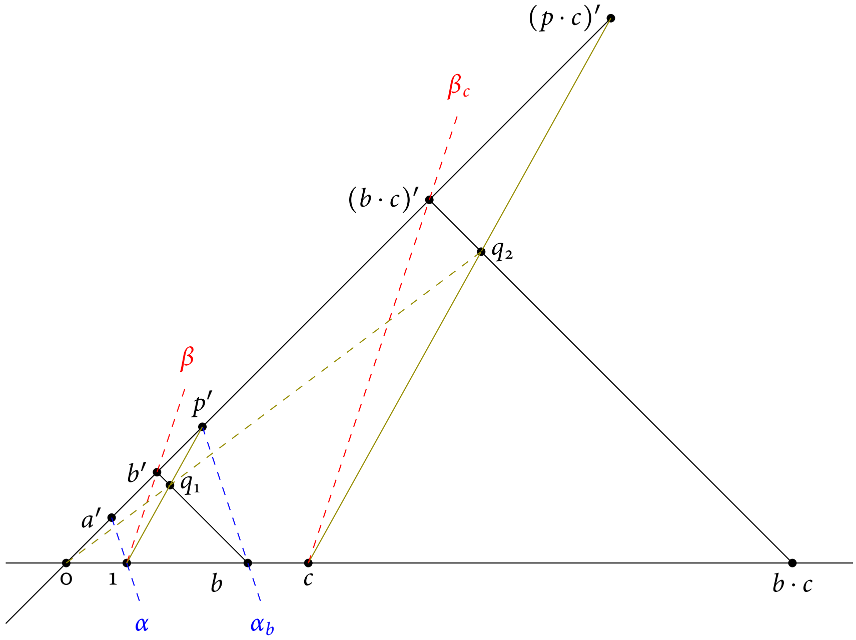
5.5.1. Exercise. *In a desarguesian affine plane, let x_1, x_2, y_1, y_2, r, s be such that $(x_1x_2) \parallel (y_1r) \parallel (y_2s)$ and $(y_1y_2) \parallel (x_1s) \parallel (x_2r)$. Prove that if $(x_1y_1) \parallel (x_2y_2)$, then $(rs) \parallel (x_1y_1)$.*



5.5.2. Exercise (multiplication)**.**

1. *Show that multiplication is well-defined.*
2. *Prove that each $a \in \ell \setminus \{0\}$ has a* left-*inverse.*
3. *Let $p = a \cdot b$; also let $q_1 = (1p') \cap n_b$ and $q_2 = (1p')_c \cap n_{b \cdot c}$. Draw and understand the following picture.*

[8] One possible first reference for near-structures is C. Weibel, Survey of non-desarguesian planes, *Not. AMS*, 54 no. 10, pp. 1294–1303, November 2007.



4. *Prove that* $0, q_1, q_2$ *are collinear. (Hint: introduce* $r = (0q_1) \cap (1p')_c$*.)*

5. *Prove that* (bp') *and* $((b \cdot c)\,(p \cdot c)')$ *are parallel.*

6. *Deduce that multiplication is associative.*

7. *Conclude that each* $a \in \setminus \{0\}$ *has a* two-sided *inverse.*

5.5.3. Exercise. *Read Chapter V of Hilbert's* Grundlagen der Geometrie *(in any language).*

5.5.4. Exercise. *Let* $J_9 = \{0, \pm 1, \pm i, \pm j, \pm k\}$ *with:*

- *addition: associative, with* $3x = 0$ *and such that* $1 + i = j$ *and* $1 - i = k$*;*
- *multiplication: like in quaternions.*

1. *Write the addition table. Prove that* J_9 *is not a skew-field.*

2. *(Harder.) Prove that* $\mathbb{P}^2(J_9)$ *(thought of as non-zero triples* $[x, y, z]$ *modulo left-action) is a projective plane.*

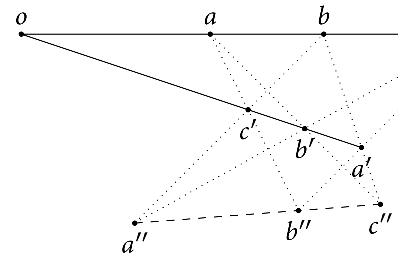
6 The Pappus property

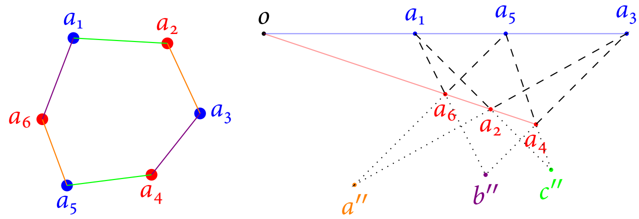
Abstract. § 6.1 defines the *Pappus property*. § 6.2 proves that 'Pappus implies Desargues'. § 6.3 returns to coordinatisation: a projective plane has the Pappus property iff it is isomorphic to $\mathbb{P}^2(\mathbb{F})$ for some *commutative* field $\mathbb{F}$.

6.1 Statement of the property

6.1.1. Definition (pappian projective plane)**.** A projective plane is *pappian* if it has the *projective Pappus property*, which is the following axiom.

> If a, b, c are collinear points and a', b', c' are collinear points (being six non-collinear points in total), then the intersection points $a'' = (cb') \cap (bc')$, $b'' = (ac') \cap (ca')$, and $c'' = (ab') \cap (ba')$ are collinear.

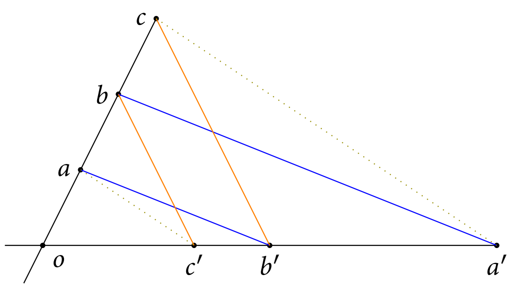




How to remember Pappus: draw two pictures, erase one. 1. Draw an ordinary hexagon. You should see 'opposite sides'. Imagine that points a_1, a_3, a_5 are on one line ℓ_1, and points a_2, a_4, a_6 are on another line ℓ_2. 2. Draw two lines ℓ_1 and ℓ_2. Hang your hexagon on ℓ_1 and ℓ_2 (do not care for convexity, not an incidence-theoretic notion) [dashed]. 3. Rematch opposite sides of the hexagon: $(a_1 a_2)$ with $(a_4 a_5)$, and so on (it is worth looking at the left picture here). Get resulting intersections [dotted]. 4. Draw the 'Pappus line' $(a'' - b'' - c'')$ and erase the left picture.

6.1.2. Definition (pappian affine plane)**.** An affine plane $\mathbb{A}$ is *pappian* if it has the *affine Pappus property*, which is the following axiom.

> Let a, b, c be collinear points and c', b', a' be collinear points (being six non-collinear points in total). If $(ab') /\!/ (ba')$ and $(c'b) /\!/ (b'c)$, then $(ac') /\!/ (ca')$.



This is just one affine Pappus configuration. Draw the other one.

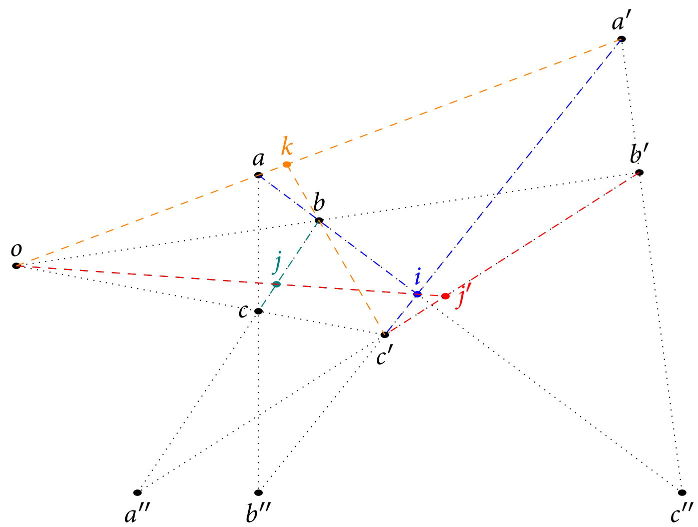
As one expects, pappianism is preserved under projectivisation/affinisation.

6.2 Pappus implies Desargues

6.2.1. Theorem (Hessenberg). *Let $\mathbb{P}$ be a projective plane. If $\mathbb{P}$ is pappian, then it is also desarguesian. The same holds of affine planes.*

Proof. Let $(o, a, b, c, a', b', c', a'', b'', c'')$ be a Desargues configuration; we aim at showing that a'', b'', c'' are collinear. We shall introduce four new points and apply Pappus' Theorem three times. Let:

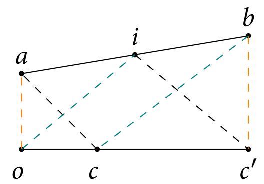
- $i = (ab) \cap (a'c')$;
- $j = (oi) \cap (bc)$;
- $j' = (oi) \cap (b'c')$;
- $k = (oa) \cap (bc')$.



Hard to read, for reference only. The initial configuration, with known collinearities; points o, i, j, j' are collinear.

Step 1. b'', j, k are collinear.

Verification. Consider the triples a, i, b and o, c, c', in this order. Notice that they are collinear by construction and the Desargues assumption.



No notion of betweenness, but order matters.

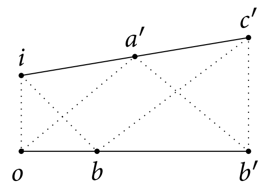
Moreover:

- $(ac) = (ab'') $ and $(ic') = (a'c') = (a'b'')$ so $(ac) \cap (ic') = b''$;
- by definition, $(oi) \cap (cb) = j$;
- by definition again, so $(ao) \cap (bc') = k$.

By Pappus' Theorem, we find the desired collinearity. ◇

Step 2. c'', j', k are collinear.

Verification. Now consider the triples i, a', c' and o, b, b'. They are also collinear by construction and the Desargues assumption.



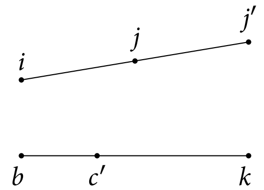
Now:

- $(ib) = (ab) = (ac'')$ and $(a'b') = (a'c'')$, so $(ib) \cap (a'b') = c''$;
- by definition, $(io) \cap (c'b') = j'$;
- $(oa') = (oa) = (ok)$ and $(bc') = (bk)$, so $(oa') \cap (bc') = k$.

Again by Pappus' Theorem, the desired points are collinear. ◇

Step 3. a'', b'', c'' are collinear.

Verification. Finally consider the triples i, j, j' and b, c', k. By construction, they are collinear.



But:

- $(ic') = (a'c') = (a'b'')$ and $(jk) = (jb'')$ by Step 1, so $(ic') \cap (jb'') = b''$;
- $(bj) = (bc) = (ba'')$ and $(c'j') = (b'c') = (b'a'')$, so $(bj) \cap (c'j') = a''$;
- $(ib) = (ab) = (ac'')$ and $(j'k) = (j'c'')$ by Step 2, so $(ib) \cap (j'k) = c''$.

Pappus' Theorem gives the conclusion. ◇

This is exactly the desired collinearity for the Desargues property. Of course we treated only the case where all points where distinct. Rigorously speaking one ought to handle the degenerate cases as well, but this is tedious.

As for the affine case, if $\mathbb{A}$ is pappian, then so is its projectivisation $\hat{\mathbb{A}}$. By the projective case, $\hat{\mathbb{A}}$ is therefore desarguesian. Now removing the line at infinity we just added, $\mathbb{A}$ is desarguesian as well. □

6.2.2. Remarks.

- Not treating annoying cases is always dangerous: Hessenberg's proof was no more fully rigorous than ours.[9]
- The converse fails: there exist desarguesian, non-pappian affine/projective planes.
- However, *for finite planes*, Desargues implies Pappus: Proposition 6.3.4 of the next subsection.

6.3 Pappus' affine and projective theorems

6.3.1. Theorem (Pappus' theorem). *Let $\mathbb{F}$ be a skew-field. Then $\mathbb{A}^2(\mathbb{F})$ is pappian iff $\mathbb{P}^2(\mathbb{F})$ is pappian iff $\mathbb{F}$ is commutative.*

Proof.

Step 1. $\mathbb{A}^2(\mathbb{F})$ is pappian iff $\mathbb{P}^2(\mathbb{F})$ is.

Verification. Add or remove lines, bearing in mind Proposition 2.3.1. ◇

The rest of the proof is easier in affine planes.

Step 2. If $\mathbb{A}^2(\mathbb{F})$ is pappian, then $\mathbb{F}$ is commutative.

Verification. Let $\lambda, \mu \in \mathbb{F}^\times$. Return to the description of affine planes $\mathbb{A}^2(\mathbb{F})$, notably Step 2 of Proposition 1.3.2: we have a description of parallelism. Let $a = (1,0)$, $b = (\lambda, 0)$, $c = (\mu \cdot \lambda, 0)$ and $c' = (0,1)$, $b' = (0,\mu)$, $a' = (0, \lambda \cdot \mu)$.

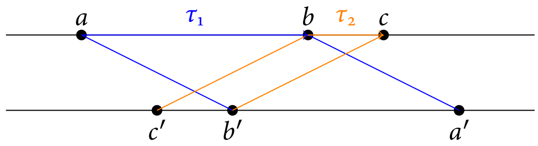
Notice that $\overrightarrow{ba'} = \lambda \cdot \overrightarrow{ab'}$ and $\overrightarrow{b'c} = \mu \cdot \overrightarrow{c'b}$, so $(ba') /\!/ (ab')$ and $(b'c) /\!/ (c'b)$. By the Pappus property, $(ac') /\!/ (ca')$. Therefore there is $\nu \in \mathbb{F}^\times$ such that $\overrightarrow{ca'} = \nu \cdot \overrightarrow{ac'}$. In coordinates, $\lambda \cdot \mu = \nu$ and $-\mu \cdot \lambda = -\nu$. This implies $\lambda\mu = \mu\lambda$, viz. commutativity of $\mathbb{F}$. ◇

Step 3. If $\mathbb{F}$ is commutative, then $\mathbb{A}^2(\mathbb{F})$ is pappian.

Verification. There are two cases.

Case 1. Suppose lines (ac) and $(c'a')$ are parallel.

[9] See A. Seidenberg. Pappus implies Desargues. Amer. Math. Monthly 83(3), 1976.



Let τ_1 be the translation mapping a to b. Then since $(b'c') = (c'a') /\!/ (ac) = (ab)$ which is the axis of the translation, $\tau_1((b'c')) = (c'a')$.

Also, $\tau_1((ab')) /\!/ (ab') /\!/ (ba')$, but $\tau_1((ab'))$ contains $\tau_1(a) = b$. So by $\mathbf{AP_2}$, one has $\tau_1((ab')) = (ba')$

In particular, $\tau_1(b') \in (c'a') \cap (ba')$ and $\tau_1(b') = a'$.

One can prove likewise that the translation τ_2 taking b to c also takes c' to b'. So the translation $\tau = \tau_2\tau_1 = \tau_1\tau_2$ takes a to c and c' to a'. As a conclusion $(ac') /\!/ \tau((ac')) = (ca')$.

We merely used commutativity of composition of translations; i.e. that $(\mathbb{F}^2; \vec{o}, +)$ is an abelian group. So far commutativity of $\mathbb{F}$ itself (i.e. of multiplication $\cdot$) was not required.

Case 2. This is more interesting. Suppose that (ac) and $(c'a')$ meet at say o.

This is the picture we gave earlier, and we shall adapt the translation argument using another tool.

Let h_1 be the homothety with centre o mapping a to b, and h_2 be the homothety with centre o mapping b to c.

Notice that:

$$b = h_1(a) \in h_1((ab')) /\!/ (ab') /\!/ (ba'),$$

so $h_1((ab')) = (ba')$. Moreover, since h_1 is a homothety, $h_1((ob')) = (ob') = (oa')$. Therefore $h_1(b') = a'$. Likewise, one proves $h_2(c') = b'$.

As a conclusion,

$$(ac') /\!/ h_2 \circ h_1((ac')) = (h_2h_1(a)\, h_2h_1(c')) = (c\, h_2h_1(c')).$$

By commutativity of $\mathbb{F}$, two homotheties with same centre commute. So actually $h_2h_1(c') = h_1h_2(c') = a'$, and now $(ac') /\!/ (ca')$, as desired. ◇

This completes the proof. □

6.3.2. Corollary (Pappus = coordinatisable in a commutative field)**.** *Let $\mathbb{P}$ be a projective plane. Then $\mathbb{P}$ is pappian iff $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$ for some* commutative *field* $\mathbb{F}$.

Proof. The converse implication is one direction of Pappus' Theorem 6.3.1. For the direct implication, let $\mathbb{P}$ be pappian. Then by Hessenberg's Theorem 6.2.1, $\mathbb{P}$ is desarguesian. Therefore by Hilbert's Theorem 3.3.1, there is a skew-field $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$. Now $\mathbb{P}^2(\mathbb{F})$ is pappian, and by Pappus' Theorem 6.3.1, $\mathbb{F}$ is commutative. □

6.3.3. Remarks.

- Our argument goes through Hessenberg's and Hilbert's theorems. I am not aware of a quicker proof directly producing coordinates from the Pappus property.

- As a consequence of Corollary 6.3.2, the theory of affine/projective planes is undecidable.[10]

Return to the implication 'Pappus implies Desargues'. The general converse fails: let $\mathbb{F}$ be any non-commutative skew-field, for instance the quaternions $\mathbb{H}$. Then $\mathbb{P}^2(\mathbb{H})$ is desarguesian but not pappian. However, in the finite case, the converse *does* hold.

6.3.4. Proposition. *Let $\mathbb{P}$ be a* finite*, desarguesian projective plane. Then $\mathbb{P}$ is pappian.*

Proof. We use Hilbert's Coordinatisation Theorem 3.3.1. Let $\mathbb{P}$ be desarguesian. Then there is a skew-field $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$. Now $\mathbb{F}$ is finite, so by Wedderburn's Theorem from the Introduction, $\mathbb{F}$ is commutative. By Pappus' Theorem 6.3.1, $\mathbb{P}$ is pappian. □

6.3.5. Remark. This argument given here is quite algebraic as it relies on the introduction of a coordinate system, and its study using Wedderburn's Theorem. The first essentially geometric proof of Proposition 6.3.4 was found by Tecklenburg.[11] But finitess is important, so number-theoretic estimates must play a role. Tecklenburg's proof relies, in the very end, on cyclotomic polynomials—as a matter of fact, the very same kind of argument that Witt so aptly introduced in order to prove... Wedderburn's Theorem.

6.4 Exercises

6.4.1. Exercise. *Draw the missing case of the affine Pappus configuration.*

6.4.2. Exercise. *Let $\mathbb{P}$ be a pappian projective plane. Show that $\check{\mathbb{P}}_\lambda$ is pappian (with respect to any line). Conversely, show that the projectivisation of a pappian affine plane remains pappian.*

7 Semi-linear automorphisms of vector spaces

Abstract. § 7.1 returns to the *general linear group* over a skew-field. § 7.2 (optional) shows how both linear automorphisms and field automorphisms induce proportionality-perserving additive automorphisms of a vector space. § 7.3 introduces *semi-linear maps* and studies the group $\Gamma\Lambda(V)$ of semi-linear automorphisms of a vector space.

Let $\mathbb{F}$ be a fixed skew-field. Eventually we want to describe automorphisms of $\mathbb{P}^2(\mathbb{F})$; in the process we must understand certain transformations of $\mathbb{F}^3$, and the general case is worth exploring.

Let V be a *left*-vector over $\mathbb{F}$. We reserve variables λ, μ for scalars, even implicitly, and x, y for vectors, dropping arrows. Since we reserve the word 'collinearity' for points in incidence geometries, we prefer to say that two non-zero vectors x, y are *proportional* if there is $\lambda \in \mathbb{F}$ with $y = \lambda x$. This is an equivalence relation. The quotient set is naturally the set of vector lines in V.

[10] J. Makowsky. Can one design a geometry engine? On the (un)decidability of certain affine Euclidean geometries. *Ann. Math. Artif. Intell.* 85 (2), pp. 259—291, 2019.

[11] H. Tecklenburg. A proof of the theorem of Pappus in finite Desarguesian affine planes, J. Geom., 30(2), pp. 172–181, 1987.

7.1 The general linear group

The definition should be known, but skew-commutativity has unexpected effects.

7.1.1. Definition (general linear group)**.** Let $\mathrm{GL}(V)$ be the group of linear automorphisms, viz. of additive automorphisms such that for all λ, v one has $g(\lambda v) = \lambda g(v)$.

These objects are also called *automorphisms of the vector space* V.

7.1.2. Remarks.

- *Scalar maps need not be linear*. More precisely let $\lambda \in \mathbb{F}$. If the space is non-trivial, then $(v \mapsto \lambda v)$ is linear iff $\lambda \in Z(\mathbb{F})$, the centre of $\mathbb{F}$.
- We do not claim that $\mathbb{F}^\times \mathrm{Id}$ is a normal subgroup of $\mathrm{GL}(V)$. *As a matter of fact,* $\mathbb{F}^\times \mathrm{Id}$ *need not be contained in* $\mathrm{GL}(V)$. (This is exactly the first remark.)
- In finite dimension, the map $(v \mapsto \lambda v)$ is *not* represented by the diagonal matrix:

$$D = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}.$$

  Indeed linear maps (in finite dimension) can be represented by matrices *acting from the right* on row vectors, and the linear map $R \mapsto R \cdot D$ is *not* the scalar action λ (the latter being 'multiplication from the left'). One may check this with quaternions.

7.2 Two constructions

This subsection serves as an introduction to the next; it may be skipped at the expense of increased conceptual difficulty. We describe two constructions of proportionality-preserving, additive automorphisms of $\mathbb{F}^3$. (This would work in any dimension but the subsection is pedagogical in nature.) *Even if* $\mathbb{F}$ *is commutative, non-linear phenomena occur.*

7.2.1. Lemma (linear automorphisms induce projective automorphisms)**.** *Any automorphism of the vector space* $\mathbb{F}^3$ *induces an automorphism of the projective plane* $\mathbb{P}^2(\mathbb{F})$.

Proof. Bear in mind that $\mathbb{P}^2(\mathbb{F}) = \mathbb{P}(\mathbb{F}^3)$ is the set of (left-)vector lines in $\mathbb{F}^3$, with lines the set of (left-)vector planes in $\mathbb{F}^3$. Any $g \in \mathrm{GL}(\mathbb{F}^3)$ preserves (left-)proportionality, so induces a map $\mathbb{P}(g): \mathbb{P}^2(V) \to \mathbb{P}^2(V)$ (in functorial notation), or $[g]: \mathbb{P}^2(V) \to \mathbb{P}^2(V)$ (in equivalence class notation). Explicitly, define $\gamma = \mathbb{P}(g) = [g]$ by letting:

$$\gamma([v]) = [g(v)],$$

which is well-defined and bijective. When acting on vector subspaces, f preserves maps vector planes to vector planes. Therefore γ maps projective lines to projective lines, and preserves incidence (inclusion). So γ is an automorphism of $\mathbb{P}^2(\mathbb{F})$. □

7.2.2. Lemma (skew-field automorphisms induce projective automorphisms)**.** *Any automorphism* $\sigma \in \mathrm{Aut}(\mathbb{F})$ *of the skew-field* $\mathbb{F}$ *induces an automorphism* $\overline{\sigma}$ *of the projective plane* $\mathbb{P}^2(\mathbb{F})$.

Proof. Let $\sigma : \mathbb{F} \to \mathbb{F}$ be a skew-field automorphism. This induces a map $V(\sigma) : \mathbb{F}^3 \to \mathbb{F}^3$ (in functorial notation). Explicitly, extend σ to a map $f = \hat{\sigma}$ on $\mathbb{F}^3$ by letting, in coordinates:

$$f((x, y, z)) = (\sigma(x), \sigma(y), \sigma(z)).$$

The map f is an additive bijection, but not a linear one: $f(\lambda \cdot v) = \sigma(\lambda) \cdot f(v)$ need not equal $\lambda \cdot f(v)$. (Technically, f is called 'σ-semi-linear'.)

Still, f preserves proportionality. Therefore (still in functorial notation), it induces a map $\mathbb{P}^2(\sigma) : \mathbb{P}^2(\mathbb{F}) \to \mathbb{P}^2(\mathbb{F})$. Explicitly, define $\overline{\sigma} = \mathbb{P}^2(\sigma) = \mathbb{P}(f) = [f]$ by letting:

$$\overline{\sigma}([v]) = [f(v)].$$

Now $\overline{\sigma}$ is clearly an automorphism of $\mathbb{P}^2(\mathbb{F})$. □

7.2.3. Remarks.

- In projective coordinates, $\overline{\sigma}([\lambda, \mu, \nu]) = [\sigma(\lambda), \sigma(\mu), \sigma(\nu)]$.
- The proof decomposes into two steps: first show that σ induces $f = \hat{\sigma}$ on $\mathbb{F}^3$, then quotient down to an automorphism $\overline{\sigma} = \mathbb{P}(f) = \mathbb{P}^2(\sigma)$ of $\mathbb{P}^2(\mathbb{F}) = \{\text{non-zero vectors}\}/\text{proportionality}$. This motivates the study of maps like f.

7.3 The group of semi-linear automorphisms

We study (additive) automorphisms of V which preserve proportionality. *Even in the commutative case, this is more general than* $\mathrm{GL}(V)$. Recall that $\mathrm{GL}(V)$ is the group of additive automorphisms which *commute* with the scalar action, viz. which preserve *proportions*. We weaken this condition.

7.3.1. Definition (semi-linear map)**.** Let $\sigma \in \mathrm{Aut}(\mathbb{F})$ be a field automorphism. An additive morphism $f : V \to V$ is σ*-semi-linear* if for all $\lambda \in \mathbb{F}$ and $x \in V$ one has:

$$(\forall \lambda)(\forall v)[f(\lambda v) = \sigma(\lambda) f(v)].$$

(This is defined *only* for additive maps.) The sum of semi-linear maps attached to distinct automorphisms is *not* semi-linear. This is a strong indication that we should focus on the group structure and not look for ring structures. Indeed, the composition of two semi-linear maps is a semi-linear map, for the composition of automorphisms; and the inverse of a semi-linear additive automorphism is one, for the inverse field automorphism.

7.3.2. Definition (group of semi-linear automorphisms)**.** Let $\Gamma\Lambda(V)$ be the group of semi-linear additive automorphisms of V.

7.3.3. Examples.

1. Every $g \in \mathrm{GL}(V)$ is linear, viz. $\mathrm{Id}_{\mathbb{F}}$-semi-linear, so $\mathrm{GL}(V) \leq \Gamma\Lambda(V)$. (Actually $\mathrm{GL}(V) \trianglelefteq \Gamma\Lambda(V)$ as one sees in Step 3 below.)
2. $\mathbb{F}^\times \mathrm{Id} \leq \mathrm{GL}(V)$. For $\lambda \in \mathbb{F}^\times$, let $\sigma_\lambda(\mu) = \lambda\mu\lambda^{-1}$ be conjugation by λ, a field automorphism. Then $(v \mapsto \lambda v)$ is σ_λ-semi-linear.
3. More generally, if $\sigma \in \mathrm{Aut}(\mathbb{F})$, then $\hat{\sigma}$ of Lemma 7.2.2 is σ-semi-linear.

The following reduces semi-linear, additive automorphisms of a vector space into two parts: linear and field-automorphic.

7.3.4. Proposition. *Suppose V has dimension ≥ 2. Then:*

(i) $\Gamma\Lambda(V)$, *the group of semi-linear additive automorphisms, is exactly the group of additive automorphisms preserving proportionality. (In symbols,* $\Gamma\Lambda(V) = \mathrm{Aut}(V; +, \sim)$ *where* $\sim$ *is proportionality.)*

(ii) $\Gamma\Lambda(V) \simeq \mathrm{GL}(V) \rtimes \mathrm{Aut}(\mathbb{F})$.

(iii) Every $f \in \Gamma\Lambda(V)$ *decomposes in a unique way as* $f = g \circ \hat{\sigma}$ *where* $g \in \mathrm{GL}(V)$ *and* $\hat{\sigma}$ *is induced by some* σ $\mathrm{Aut}(\mathbb{F})$.

7.3.5. Remarks.

- In dimension 1, *every* additive automorphism preserves proportionality so the proposition fails.
- One may not drop additivity: otherwise, stabilise each line and permute inside each line, independently. This preserves proportionality.
- Even in the commutative case, $\mathrm{GL}(V)$ may be proper in $\Gamma\Lambda(V)$: it only depends on $\mathrm{Aut}(\mathbb{F})$. (The phenomenon went unnoticed because $\mathrm{Aut}(\mathbb{R}) = \{\mathrm{Id}\}$.)

Proof.

Step 1. $\Gamma\Lambda(V)$ preserves proportionality.

Verification. If $f \in \Gamma\Lambda(V)$ is σ-semi-linear, then for given λ and v, one has $f(\lambda v) = \sigma(\lambda) f(v)$. So proportionality is preserved. ◇

Step 2. An additive automorphism preserving proportionality is semi-linear.

Verification. Let $f\colon V \to V$ be an additive automorphism preserving proportionality. For $\lambda \in \mathbb{F}^\times$ and $x \in V \setminus \{0\}$ there is $\sigma_x(\lambda) \in \mathbb{F}$ with $f(\lambda x) = \sigma_x(\lambda) f(x)$. Also define $\sigma_x(0) = 0$.

If x and y are (left-)independent vectors, then so are $f(x)$ and $f(y)$. Now:

$$\begin{aligned}\sigma_x(\lambda) f(x) + \sigma_y(\lambda) f(y) &= f(\lambda x) + f(\lambda y)\\ &= f(\lambda(x+y))\\ &= \sigma_{x+y}(\lambda) f(x) + \sigma_{x+y}(\lambda) f(y).\end{aligned}$$

Using independence, we see that $\sigma_x(\lambda) = \sigma_{x+y}(\lambda) = \sigma_y(\lambda)$ does not depend on x; we write it σ. (The argument fails if $\dim V \leq 1$.) By construction, $\sigma(1) = 1$.

Now:

$$\begin{aligned}\sigma(\lambda+\mu) f(x) &= f((\lambda+\mu)x)\\ &= f(\lambda x) + f(\mu x)\\ &= (\sigma(\lambda) + \sigma(\mu)) f(x),\end{aligned}$$

and:

$$\begin{aligned}\sigma(\lambda\cdot\mu)f(x) &= f((\lambda\cdot\mu)x)\\ &= f(\lambda\cdot(\mu x))\\ &= \sigma(\lambda)f(\mu(x))\\ &= \sigma(\lambda)\cdot(\sigma(\mu)x).\end{aligned}$$

These imply additivity and multiplicativity, respectively. So $\sigma\colon\mathbb{F}\to\mathbb{F}$ is a field embedding. Finally consider f^{-1}, which preserves proportionality: the same applies, so there is a field embedding $\tau\colon\mathbb{F}\to\mathbb{F}$ with $f^{-1}(\lambda x)=\tau(\lambda)f^{-1}(x)$. Therefore for arbitrary λ:

$$\lambda x = f(f^{-1}(\lambda x)) = f(\tau(\lambda)f^{-1}x) = \sigma(\tau(\lambda))x,$$

which proves that σ is onto, hence a field automorphism. ◇

This proves (i). We now embed $\mathrm{Aut}(\mathbb{F})$ into $\Gamma\Lambda(V)$. For $\sigma\in\mathrm{Aut}(\mathbb{F})$ let $\hat{\sigma}$ be the automorphism defined as follows. Fix a (left-)basis $\mathcal{B}=\{e_i : i\in I\}$ of V. For $x=\sum\lambda_i e_i$ let:

$$\hat{\sigma}(x)=\sum\sigma(\lambda_i)e_i.$$

This is a σ-semi-linear automorphism. Let $\hat{\Sigma}$ be the set of operators $\hat{\sigma}$; clearly $\mathrm{Aut}(\mathbb{F})\simeq\hat{\Sigma}\leq\Gamma\Lambda(V)$. (The construction depens on the choice of $\mathcal{B}$.)

Step 3. $\Gamma\Lambda(V)=\mathrm{GL}(V)\rtimes\hat{\Sigma}$.

Verification. We prove $\mathrm{GL}(V)\trianglelefteq\Gamma\Lambda(V)$. If $g\in\mathrm{GL}(V)$ and $f\in\Gamma\Lambda(V)$ is σ-semi-linear, then for arbitrary λ, x one has:

$$\begin{aligned}(f^{-1}gf)(\lambda x) &= f^{-1}\circ g(\sigma(\lambda)f(x)) && \text{by } \sigma\text{-semi-linearity of } f\\ &= f^{-1}(\sigma(\lambda)g(f(x))) && \text{by linearity of } g\\ &= \sigma^{-1}(\sigma(\lambda))(f^{-1}gf)(x) && \text{by } \sigma^{-1}\text{-semi-linearity of } f^{-1},\end{aligned}$$

so $f^{-1}gf$ is linear.

We prove $\mathrm{GL}(V)\cap\hat{\Sigma}=\{\mathrm{Id}\}$. If $f\in\mathrm{GL}(V)\cap\hat{\Sigma}$, then $f=\hat{\sigma}$ is linear, so $\sigma=\mathrm{Id}$ and $f=\hat{\mathrm{Id}}=\mathrm{Id}$.

It remains to show that $\mathrm{GL}(V)$ and $\hat{\Sigma}$ generate $\Gamma\Lambda(V)$. Let $f\in\Gamma\Lambda(V)$; say f is σ-semi-linear. We give two arguments.

- First option. Let $g=f\circ\hat{\sigma}^{-1}$, an additive automorphism which is $\sigma\circ\sigma^{-1}$-semi-linear. This means that g is linear and we are done.
- Second option. The image $f(\mathcal{B})$ is a basis and $\mathrm{GL}(V)$ acts transitively on bases, so up to composing on the left with some $g\in\mathrm{GL}(V)$ we may assume that $f\in\Gamma\Lambda(V)$ fixes $\mathcal{B}$ pointwise. But f is σ-semi-linear, so it coincides with $\hat{\sigma}$ everywhere; hence $f\in\hat{\Sigma}$ and we are done.

◇

So $\Gamma\Lambda(V)\simeq\mathrm{GL}(V)\rtimes\mathrm{Aut}(\mathbb{F})$, proving (ii). And (iii) is a mere rephrasing. □

We now have full understanding of semi-linear automorphisms of V.

7.4 Exercises

7.4.1. Exercise. *Let $\mathbb{F}$ be a skew-field and V be a non-trivial left-vector space. Prove that* $Z(\mathrm{GL}(V)) = Z(\mathbb{F}^\times)\,\mathrm{Id}$.

8 Automorphisms of projective planes

Abstract. Let $\mathbb{P}$ be a projective plane and $G = \mathrm{Aut}(\mathbb{P})$. There are two cases:

- $\mathbb{P}$ is coordinatisable (= desarguesian): we know a lot about G;
- $\mathbb{P}$ is non-coordinatisable (= non-desarguesian): nothing general can be said about G as an abstract group.

It must be borne in mind that the present section deals mostly with coordinatisable (= desarguesian) projective planes.

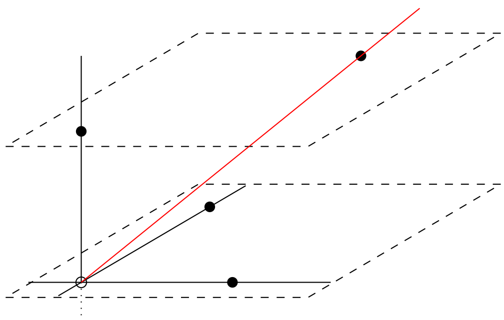
8.1 A lemma on coordinates

The following technical lemma confirms that the coordinate system is 'built in' the incidence geometry $\mathbb{P}^2(\mathbb{F})$. Though it naturally belongs to the flow of the argument of Theorem 8.2.1 we present it here as its proof is long.[12]

8.1.1. Lemma. *If $\alpha \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$ fixes $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$, then it is of the form $\mathbb{P}^2(\sigma) = [\hat{\sigma}]$ for some skew-field automorphism σ.*

Proof not covered in class.

Proof.



Consider the following variable points for $\lambda, \mu \in \mathbb{F}$:

$$q_\lambda = [\lambda, 0, 1],$$
$$r_\mu = [0, \mu, 1],$$
$$s_{\lambda,\mu} = [\lambda, \mu, 1].$$

Note that $(\lambda, 0, 1) = \lambda e_1 + e_3 \in \langle e_1, e_3\rangle$, so $q_\lambda \in (p_1 p_3)$; moreover $q_\lambda \neq p_1$. Likewise $r_\mu \in (p_2 p_3) \setminus \{p_1\}$. Finally $(p_1 r_\mu) \cap (p_2 q_\lambda) = \{s_{\lambda,\mu}\}$.

[12] I follow A. Keedwell, Self-collineations of desarguesian projective planes, *Am. Math. Monthly*, 82 no. 1, pp. 59–63, January 1975.

Step 1. There is a function $\sigma: \mathbb{F} \to \mathbb{F}$ such that for all $\lambda \in \mathbb{F}$:

$$\varphi([\lambda, 0, 1]) = [\sigma(\lambda), 0, 1].$$

Verification. Since φ fixes each p_i, it stabilises each line through two of them. Therefore there is a map $\sigma: \mathbb{F} \to \mathbb{F}$ with $\varphi(q_\lambda) = q_{\sigma(\lambda)}$. (We cannot prove yet that it is a field automorphism.) In coordinates:

$$\varphi([\lambda, 0, 1]) = [\sigma(\lambda), 0, 1],$$

as claimed. ◇

Step 2. One also has $\varphi([\lambda, \mu, 1]) = [\sigma(\lambda), \sigma(\mu), 1]$. Moreover, $\sigma(1) = 1$.

Verification. In a similar way there is $\tau: \mathbb{F} \to \mathbb{F}$ with $\varphi([0, \mu, 1]) = [0, \tau(\mu), 1]$. But then:

$$\begin{aligned}\varphi(s_{\lambda,\mu}) &= \varphi((p_1 r_\mu) \cap (p_2 q_\lambda)) \\ &= (p_1 r_{\tau(\mu)}) \cap (p_2 q_{\sigma(\lambda)}) \\ &= s_{\sigma(\lambda),\tau(\mu)},\end{aligned}$$

in coordinates: $\varphi([\lambda, \mu, 1]) = [\sigma(\lambda), \tau(\mu), 1]$.

But when $\lambda = \mu$ the point $r_{\lambda,\lambda}$ is on $(p_1 p_4)$, which is stabilised by φ. So $\varphi(r_{\lambda,\lambda})$ is some $r_{\nu,\nu}$. In coordinates:

$$[\sigma(\lambda), \tau(\lambda), 1] = [\nu, \nu, 1].$$

This proves $\sigma(\lambda) = \tau(\lambda)$: functions are equal.

It follows $\varphi([\lambda, \mu, 1]) = [\sigma(\lambda), \sigma(\mu), 1]$. And clearly, $\sigma(1) = 1$. ◇

Step 3. One even has $\varphi([1, \lambda, \mu]) = [1, \sigma(\lambda), \sigma(\mu)]$.

Verification. Likewise, there is $\sigma': \mathbb{F} \to \mathbb{F}$ such that:

$$\varphi([1, \lambda, \mu]) = [1, \sigma'(\lambda), \sigma'(\mu)],$$

and $\sigma'(1) = 1$. But then:

$$[1, \sigma(\lambda), 1] = \varphi([1, \lambda, 1]) = [1, \sigma'(\lambda), 1],$$

so $\sigma' = \sigma$. ◇

Step 4. σ is a field automorphism and $\varphi([\lambda, \mu, \nu]) = [\sigma(\lambda), \sigma(\mu), \sigma(\nu)]$.

Verification. By the above,

$$\begin{aligned}[1,\sigma(\lambda^{-1}),\sigma(\lambda^{-1})] &= \varphi([1,\lambda^{-1},\lambda^{-1}])\\ &= \varphi([\lambda,1,1])\\ &= [\sigma(\lambda),1,1]\\ &= [1,\sigma(\lambda)^{-1},\sigma(\lambda)^{-1}],\end{aligned}$$

proving $\sigma(\lambda^{-1}) = \sigma(\lambda)^{-1}$. Therefore:

$$\begin{aligned}[\sigma(\lambda),\sigma(\lambda\mu),1] &= \varphi([\lambda,\lambda\mu,1])\\ &= \varphi([1,\mu,\lambda^{-1}])\\ &= [1,\sigma(\mu),\sigma(\lambda^{-1})]\\ &= [\sigma(\lambda),\sigma(\lambda)\sigma(\mu),1],\end{aligned}$$

proving multiplicativity.

Hence:

$$\begin{aligned}\varphi([\lambda,\mu,\nu]) &= \varphi([1,\lambda^{-1}\mu,\lambda^{-1}\nu])\\ &= [1,\sigma(\lambda^{-1})\sigma(\mu),\sigma(\lambda^{-1})\sigma(\nu)]\\ &= [\sigma(\lambda),\sigma(\mu),\sigma(\nu)].\end{aligned}$$

This and $\sigma(1) = 1$ implies that φ fixes $[1,0,1]$ and $[0,1,1]$. It therefore stabilises the line through them, which is exactly $\{[\lambda,\mu,\lambda+\mu] : \lambda,\mu \in \mathbb{F}\}$. Therefore $\varphi([\lambda,\mu,\lambda+\mu])$ has the same form, which proves additivity of σ. Surjectivity is no issue; it is a field automorphism. ◇

It is now clear that φ is induced by $\sigma \in \mathrm{Aut}(\mathbb{F})$, in our notation $\varphi = \overline{\sigma} = \mathbb{P}^2(\sigma)$. □

8.2 The projective version

We now let $V = \mathbb{F}^3$ and let $\Gamma\Lambda(\mathbb{F}^3)$ act on $\mathbb{F}^3$; since it preserves proportionality, it induces an action on $\mathbb{P}^2(\mathbb{F})$. For $f \in \Gamma\Lambda(\mathbb{F}^3)$ we let $[f]$ be the induced map. If $p \in \mathbb{P}^2(\mathbb{F})$ has projective coordinates $p = [x,y,z]$, viz. if $p = [v]$ with $v = (x,y,z)$, then:

$$[f](p) = [f(v)].$$

8.2.1. Theorem.

(i) The action of $\Gamma\Lambda(\mathbb{F}^3)$ *on* $\mathbb{P}^2(\mathbb{F})$ *induces a group homomorphism* $\Gamma\Lambda(\mathbb{F}^3) \to \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$.

(ii) The kernel of the action is $\mathbb{F}^\times \mathrm{Id}$*, the group of scalar maps.*

(iii) The action induces $\mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$.

(iv) Every $\alpha \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$ *factors uniquely as* $\alpha = \gamma \circ \beta$ *with* $\gamma = [g]$ *and* $\beta = [\hat{\sigma}]$ *for some* $g \in \mathrm{GL}(V)$ *and* $\sigma \in \mathrm{Aut}(\mathbb{F})$.

8.2.2. Remarks.

- Have another look at Remark 7.1.2; $\mathbb{F}^\times \mathrm{Id}$ is a normal subgroup of $\Gamma\Lambda(\mathbb{F}^3)$ be need not be contained in $\mathrm{GL}(\mathbb{F}^3)$.

- In (iv), γ and β are well-defined but g is not; g is unique only up to $Z(\mathbb{F}^\times)\,\mathrm{Id} = Z(\mathrm{GL}(\mathbb{F}^3))$.

Proof. Let $f \in \Gamma\Lambda(V)$. Then $f\colon \mathbb{F}^3 \to \mathbb{F}^3$ maps vector planes to vector planes. Therefore, the induced map $Pf\colon \mathbb{P}^2(\mathbb{F}) \to \mathbb{P}^2(\mathbb{F})$ maps projective lines to projective lines. It is obviously incidence preserving. Therefore $Pf \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$.

Step 1. The kernel of the action is $\mathbb{F}^\times\,\mathrm{Id}$.

Verification. Suppose $f \in \Gamma\Lambda(V)$ acts trivially on $\mathbb{P}^2(\mathbb{F})$. Then for any $x \in \mathbb{F}^3$ there is $\lambda_x \in \mathbb{F}$ with $f(x) = \lambda_x x$. Using the ordinary tricks, λ_x does not depend on x. So $f = \lambda\,\mathrm{Id} \in \mathbb{F}^\times\,\mathrm{Id}$. (Recall that we do *not* assert linearity of these maps: neither did we suppose linearity of f.) The converse is obvious. ◇

To prove surjectivity is not as easy. Let X be the set of 4-tuples of points in $\mathbb{P}^2(\mathbb{F})$ no three of which are collinear.

8.2.3. Lemma. *The group* $\mathrm{PGL}_3(\mathbb{F})$ *acts transitively on X, viz. if* (p_1, p_2, p_3, p_4) *and* (q_1, q_2, q_3, q_4) *are in X, then there is* $\varphi \in \mathrm{GL}_3(\mathbb{F})$ *such that* $\psi = \mathbb{P}(\varphi)$ *maps each* p_i *to* q_i.

Proof. As always in group theory, we may choose p_1, p_2, p_3, p_4. Let $e_1 = (1, 0, 0) \in \mathbb{F}^3$ and $p_1 = [e_1]$; define e_2, e_3, p_2, p_3 likewise. Also let $e_4 = (1, 1, 1)$; notice that any three of the e_i's generate $\mathbb{F}^3$, so any three of the p_i's are non-collinear in $\mathbb{P}(\mathbb{F}^3) = \mathbb{P}^2(\mathbb{F})$. It is therefore enough to prove the lemma with $p_i = [e_i]$.

Write each q_i in projective coordinates, say $q_i = [f_i]$ with $f_i = (a_{i,1}, a_{i,2}, a_{i,3}) \in \mathbb{F}^3$. Let:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Introduce unknown scalars $\lambda_1, \lambda_2, \lambda_3$ and consider the matrix:

$$M_\lambda = \begin{pmatrix} \lambda_1 a_{1,1} & \lambda_1 a_{1,2} & \lambda_1 a_{1,3} \\ \lambda_2 a_{2,1} & \lambda_2 a_{2,2} & \lambda_2 a_{2,3} \\ \lambda_3 a_{3,1} & \lambda_3 a_{3,2} & \lambda_3 a_{3,3} \end{pmatrix}.$$

Regarldess of the λ_i's, one has $(1, 0, 0) \cdot M_\lambda = \lambda_1 \cdot (a_{1,1}, a_{1,2}, a_{1,3}) = \lambda_1 \cdot f_1$, so $p_1 \cdot [M_\lambda] = q_1$ and likewise for indices 2 and 3. So it suffices to find $\lambda_1, \lambda_2, \lambda_3$ such that $p_4 \cdot [M_\lambda] = q_4$.

Now observe that:

$$(1, 1, 1) \cdot M_\lambda = (\lambda_1, \lambda_2, \lambda_3) \cdot A.$$

Therefore, $p_4 \cdot [M_\lambda] = [\lambda_1, \lambda_2, \lambda_3] \cdot [A]$. But A maps the independent triple e_1, e_2, e_3 to the independent triple f_1, f_2, f_3. So it is left-invertible: for any row R there is a row S such that $S \cdot A = R$. We apply this with $R = f_4$; take suitable S and write it as $S = (\lambda_1, \lambda_2, \lambda_3)$. Then $p_4 \cdot [M_\lambda] = [S \cdot A] = [R] = [f_4] = q_4$, we are done. □

Step 1. $\Gamma\Lambda(\mathbb{F}^3)$ covers $\mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$.

Verification. Let $\alpha \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$. Consider the four points $p_1 = [1,0,0]$, $p_2 = [0,1,0]$, $p_3 = [0,0,1]$, and $p_4 = [1,1,1]$. No three of them are collinear, meaning $(p_1, p_2, p_3, p_4) \in X$. Let $q_i = \alpha(p_i)$; no three of these are collinear so $(q_1, q_2, q_3, q_4) \in X$. By Lemma 8.2.3 there is $\gamma = \mathbb{P}(g)$ doing $\gamma(p_i) = q_i$. Up to considering $\gamma^{-1} \circ \alpha$, we may therefore suppose that α fixes each p_i. Then by Lemma 8.1.1, there is a skew-field automorphism σ such that $\alpha = [\hat{\sigma}]$. So α is induced by $\Gamma\Lambda(V)$. ◇

Step 2. Proof of (iv).

Verification. We use Proposition 7.3.4. Let $\alpha \in \mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$. Then there is $f \in \Gamma\Lambda(\mathbb{F}^3)$ with $\alpha = [f]$. Now $f = g \circ \hat{\sigma}$ with $g \in \mathrm{GL}(\mathbb{F}^3)$ and $\sigma \in \mathrm{Aut}(\mathbb{F})$. Then $\alpha = [f] = [g] \circ [\hat{\sigma}]$ has the desired form.

It remains to check uniqueness. If also $\alpha = \gamma_2 \circ \beta_2$ with $\gamma_2 = [g_2]$ and $\beta_2 = [\hat{\sigma}_2]$, then $[g_2^{-1} g] = [\hat{\sigma}_2 \hat{\sigma}^{-1}]$, so letting $h = g_2^{-1} g \in \mathrm{GL}(\mathbb{F}^3)$ and $\tau = \sigma_2 \sigma^{-1} \in \mathrm{Aut}(\mathbb{F})$, we have $[h] = [\hat{\tau}]$. For e_i in the chosen basis, there is λ_i such that $h(e_i) = \lambda_i e_i$; by the usual tricks λ does not depend on i. Now $h = \lambda\,\mathrm{Id}$ is linear, so $\lambda \in Z(\mathbb{F}^\times)$. So $g = \lambda g_2$ and therefore $\gamma_2 = \gamma$; we are done. ◇

This gives a full description of $\mathrm{Aut}(\mathbb{P}^2(\mathbb{F}))$. □

8.2.4. Remarks (the 'fundamental theorem')**.**

- Some sources tend to forget automorphisms of type $\mathbb{P}^2(\sigma)$ because their reference field, the real field $\mathbb{R}$, has no non-trivial automorphisms.
- More in general: if V is a vector space of dimension at least 3, let $\mathbb{P}(V)$ be the set of its vector lines. Any automorphism of $\mathbb{P}(V)$ preserving collinearity of projective points is of the form $\mathbb{P}(f)$ for some semi-linear automorphism f of V. There are elementary proofs avoiding coordinates.[13]
- By calling this (or similar) result 'the fundamental theorem of projective geometry', one focuses only on *coordinatisable* structures. This terminology hides the theory of non-desarguesian projective planes and should be avoided.

8.3 The general case

Nothing general can be said about abstract projective planes. As a matter of fact, Mendelsohn proved the following difficult result.[14]

8.3.1. Theorem (Mendelsohn)**.** *Let G be a group, and let κ be any infinite cardinal. Then there exists a projective plane* $\mathbb{P} = (\mathcal{P}, \mathcal{L}, I)$ *such that* $\mathrm{Aut}(\mathbb{P}) \simeq G$ *and* $\mathrm{card}\,\mathbb{P} = \max(\kappa, \mathrm{card}\,G)$.

(Notice that nothing is said about *finite* projective with given *finite* automorphism group. Determining for which finite groups G there is finite $\mathbb{P}$ with $\mathrm{Aut}(\mathbb{P}) \simeq G$ is likely to be extremely difficult if not impossible.)

[13] C.-A. Faure, An elementary proof of the fundamental theorem of projective geometry, *Geometriae Dedicata* 90, pp. 145–150, 2002.

[14] E. Mendelsohn. Every group is the collineation group of some projective plane. *J. Geom.*, 2, pp. 97–106, 1972.

However, there is another difficult result, known only for finite projective planes.[15]

8.3.2. Theorem (Ostrom-Wagner). *Let $\mathbb{P}$ be a finite projective plane. If* $\mathrm{Aut}(\mathbb{P})$ *is doubly transitive, then $\mathbb{P}$ is coordinatisable.*

8.3.3. Remark. The suitable model-theoretic analogue is a research question.

Experts in projective planes tend to use the word 'collineation' for an incidence-preserving (viz. collinearity-preserving) morphism. We do not see the point of introducing specific terminology where general algebraic language is sufficiently illuminating.

8.4 Exercises

8.4.1. Exercise. *Let $\mathbb{F}$ be a skew-field.*

1. *Show that:*
$$[x : y] \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [xa + yc : xb + yd]$$
defines an action of $\mathrm{PGL}_2(\mathbb{F})$ *on* $\mathbb{F} \cup \{\infty\}$.
2. *Prove that the action is always* 3*-transitive.*
3. *Prove that it is* sharply 3*-transitive iff $\mathbb{F}$ a commutative field.*
4. *Let $\mathbb{F}$ be a skew-field such that any two non-zero elements are conjugate in $\mathbb{F}^\times$ (these do exist, but it is hard[16]). Prove that in this case, the action is even* 4*-transitive.*

9 Group-theoretic analysis of Desargues and Pappus

Abstract. Interpretations of the Desargues and Pappus properties in group-theoretic terms. § 9.1 interprets the Desargues propery as transitivity of a certain family of group actions. § 9.2 introduces partial maps called *projectivities*; the group of self-projectivities of a line is 3-transitive. § 9.3 characterises the Pappus property as *sharp* 3-transitivity of the latter.

9.1 Group-theoretic interpretation of the Desargues property

9.1.1. Definition. Let $\mathbb{P}$ be a projective plane; let p be a point and ℓ be a line. Let $G_{p,\ell}$ be the group of automorphisms of $\mathbb{P}$ which:

- stabilise (setwise) each line through p,
- and fix (pointwise) ℓ.

The definition supposes neither $p \in \ell$ nor $p \in \ell$.

9.1.2. Proposition. *Let $\mathbb{P}$ be a projective plane. Then $\mathbb{P}$ is desarguesian iff for all choices of p and ℓ one has:*

[15] T. Ostrom, A. Wagner, On projective and affine planes with transitive collineation groups, *Math. Z.*, 71, pp. 186–199, 1959.
[16] Exercise 3 p. 238 in: P. Cohn, P., *Skew fields, theory of general division rings*, Encyclopedia of Mathematics and its Applications, vol. 57, Cambridge University Press, 1977.

for each line m through p, the group $G_{p,\ell}$ is transitive on $m \setminus (\{p\} \cup \ell)$.

Proof.

Step 1. Direct implication.

Verification. Suppose transitivity. Let $(o; a, b, c; a', b', c; a'', b'', c'')$ be a (non-degenerate) Desargues configuration. We want to prove that a'', b'', c'' are collinear. Let $p = o$ and $\ell = (a''b'')$; we have to show that $c'' \in \ell$.

By assumption there is $g \in G_{p,\ell}$ taking a to a'. This g fixes $(a''b'')$ pointwise, so it fixes b''. Therefore it takes (ab'') to $(a'b'')$. It also fixes (cc') setwise, as a line through o. Now $(cc') \cap (ab'') = c$, while $(cc') \cap (a'b'') = c'$, so $g(c) = c'$. The same method proves $g(b) = b'$. Let $x = (ab) \cap \ell$. By definition, $g(x) = x$. Therefore $x = g(x) \in g((ab)) = (a'b')$. So (ab) and $(a'b')$ meet on ℓ. But the intersection point is c'', by definition. ◇

Step 2. Converse implication.

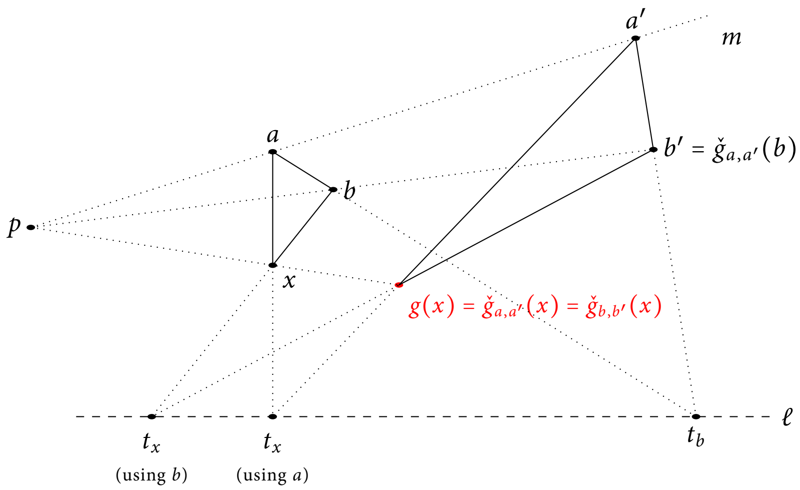
Verification. We only sketch the idea as it plays a key role in Artin's proof of Hilbert's Coordinatisation Theorem.

Suppose desarguesianity. Take p, ℓ, then m through p and $a, a' \in m \setminus (\{p\} \cup \ell)$. If $a' = a$ then $g = \mathrm{Id}$ does; so suppose not, so that $m = (aa')$. We define a *partial* map $\check{g}_{a,a'}: \mathbb{P} \setminus (\ell \cup m \cup \{p\}) \to \mathbb{P}$ as follows.

> Let $x \in \mathbb{P} \setminus (\ell \cup m \cup \{p\})$, compute $t_x = (ax) \cap \ell$ then $y_x = (a't_x) \cap (px)$, and let $\hat{g}_{a,a'}(x) = y_x$.

We extend this 'generically given function' to all of $\mathbb{P}$ by trading the base pair (a, a') for another one. Let b be any point in $\mathbb{P} \setminus (\ell \cup m \cup \{p\})$ and $b' = \check{g}_{a,a'}(b)$. Since $a \neq a'$, one has $b \neq b'$; so let $m' = (bb')$. Now for $x \in \mathbb{P} \setminus (\ell \cup m' \cup \{p\})$, let $g(x) = \hat{g}_{b,b'}(x)$.

This is well-defined. Indeed, if $x \notin (\ell \cup m \cup m' \cup \{p\})$, then $\hat{g}_{a,a'}(x) = \hat{g}_{b,b'}(x)$ reduces to a Desargues property, which holds.



There are two ways to construct $g(x)$; they agree.
(Whether $p \in \ell$ or $p \notin \ell$ remains irrelevant.)

Notice that g is now also defined on m, and maps a to a'. Complete it by letting g fix p and ℓ pointwise. By construction, $g \in G_{p,\ell}$ sends a to a'. It is a collinearity-preserving bijection. ◇

The equivalence is proved. □

9.1.3. Remarks.

- The action is then *sharply* transitive as there is a unique $g \in G_{p,\ell}$ taking a to a'.

  This can be seen during the proof (at each step, there is no choice), or after the proof. Indeed, once $\mathbb{P}$ is known to be desarguesian, it follows $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$ for some skew-field $\mathbb{F}$. Removing ℓ has the effect of taking us to the affine plane $\mathbb{A}^2({}_bF)$.

  Then the action of (the restriction of) $G_{p,\ell}$ is either that of translations parallel to ℓ ($p \in \ell$) or of homotheties with centre p. In either case, there is only one possibility for g: this is sharp transitivity.

- There is more group theory to do here, notably in the finite case. For example, Gleason[17] proved the following.

  Let $\mathbb{P}$ be a *finite* projective plane. Suppose that for all p, ℓ with $p \in \ell$, one has $G_{p,\ell} \neq \{1\}$. Then $\mathbb{P}$ is desarguesian.

  André[18] proved a similar result, quantifying over pairs (p, ℓ) with $p \notin \ell$.

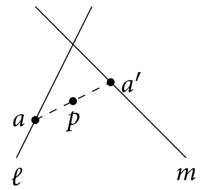
9.2 Projectivities

This subsection and the next deal with *local* functions, viz. maps not defined on the whole of a projective plane. We introduce a relevant group for the local analysis of projective planes. Its elements are called *self-projectivities*. Defining it requires some terminology.

[17] A. Gleason, Finite Fano Planes, *Amer. Jour. Math*, 78 no. 4, pp. 797–807, October 1956

[18] J. André, Über Perspektivitäten in endlichen projektiven Ebenen, *Arch. Math.* 6, pp. 29–32, 1954.

9.2.1. Definition (perspectivity)**.** Let $\ell \neq m$ be two lines and $p \notin \ell \cup m$. The *perspectivity* $\ell \underset{p}{\longrightarrow} m$ is the map taking $a \in \ell$ to $a' = (ap) \cap m$.



9.2.2. Definition (projectivity)**.** A *projectivity* from ℓ to m is any composition of perspectivities $\ell \underset{a_1}{\longrightarrow} \ell_1 \dots \underset{a_n}{\longrightarrow} \ell_n = m$.

If $m = \ell$ the map is called a *self-projectivity* of ℓ.

9.2.3. Notation (group of self-projectivities of a line)**.** Let $\mathrm{Proj}(\ell)$ be the group of self-projectivities of ℓ.

Let ℓ, m be two lines. Since there is a perspectivity between, $\mathrm{Proj}(\ell) \simeq \mathrm{Proj}(m)$. So the object does not depend on the line; it captures the local behaviour 'anywhere'.

9.2.4. Remark. I do not know whether self-projectivities always extend to automorphisms of $\mathbb{P}$.

9.2.5. Remarks (the projective *line*)**.** In this remark we assume the presence of a coordinate system, viz. $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$ for a skew-field $\mathbb{F}$.

- Since $\mathrm{Proj}(\ell)$ does not depend on ℓ, we may focus on the line with equation $z = 0$, viz. $\ell = \{[x, y, 0] : x, y \in \mathbb{F}\}$. Hence $\ell \in \mathbb{P}^2(\mathbb{F})$ can be represented as $\mathbb{P}^1(\mathbb{F}) = \{[x : y]; (x, y) \in \mathbb{F}^2 \setminus \{(0, 0)\}\} = \mathbb{F} \cup \{\infty\}$.
- Be careful that as an incidence structure, $\mathbb{P}^1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$ consists of only one line. So as an incidence structure, $\mathrm{Aut}(\mathbb{P}^1(\mathbb{F})) = \mathrm{Sym}(\mathbb{P}^1(\mathbb{F})) = \mathrm{Sym}(\mathbb{F} \cup \{\infty\})$. (Geometers usually consider more structure on $\mathbb{P}^1(\mathbb{F})$ such as the cross-ratio, which makes sense over commutative fields.)

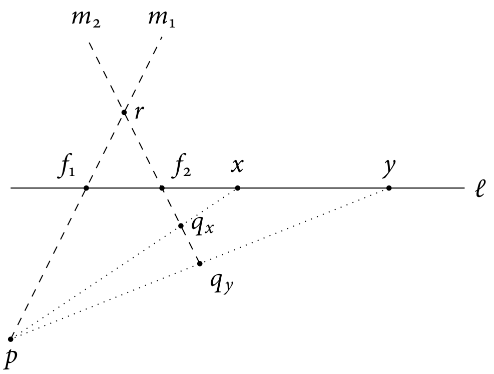
  We return to $\mathbb{P}^1(\mathbb{F})$, equipped with the notion of self-projectivites coming from its embedding *as a line in the plane*.
- Every element of $\mathrm{PGL}_2(\mathbb{F})$ can be written as a self-projectivity. Now, every self-projectivity of $\ell \in \mathbb{P}^2(\mathbb{F})$ can (in theory) be computed in coordinates, but it is unclear to me whether one remains in $\mathrm{PGL}_2(\mathbb{F})$. (There could be 'inner field automorphisms', viz. automorphisms of $\mathbb{F}$ of the form $x \mapsto \lambda^{-1} x \lambda$.)
- However, *if* $\mathbb{F}$ *is commutative*, then every self-projectivity comes from an element of $\mathrm{PGL}_2(\mathbb{F})$. Therefore $\mathrm{Proj}(\ell) \simeq \mathrm{PGL}_2(\mathbb{F})$. The action of $\mathrm{PGL}_2(\mathbb{F})$ on $\mathbb{P}^1(\mathbb{F})$ is known: see exercise 8.4.1.

One tends to avoid speaking of 'the projective line' when there is no coordinate system behind.

9.2.6. Proposition. *Let* $\mathbb{P}$ *be any projective plane with line* ℓ*. Then* $\mathrm{Proj}(\ell)$ *acts* 3*-transitively on* ℓ*.*

Proof. Let f_1, f_2 be two distinct points on ℓ: we show that the stabiliser of f_1 and f_2 in $\text{Proj}(\ell)$ acts transitively on $\ell \smallsetminus \{f_1, f_2\}$. Let x and y be points there.

Let $m_1 \neq \ell$ be a line through f_1, and $m_2 \neq \ell$ be a line through f_2. Choose any $p \in m_1 \smallsetminus \ell$. Now let $q_x = (px) \cap m_2$ and $q_y = (py) \cap m_2$. Finally, let $r = m_1 \cap m_2$.



Consider the two perspectivities $\ell \underset{q_x}{\longrightarrow} m_1$ and $m_1 \underset{q_y}{\longrightarrow} \ell$. One may follow on the picture that they have the following effect:

ℓ	$\underset{q_x}{\longrightarrow}$	m_1	$\underset{q_y}{\longrightarrow}$	ℓ
f_1	$\mapsto$	f_1	$\mapsto$	f_1
f_2	$\mapsto$	r	$\mapsto$	f_2
x	$\mapsto$	p	$\mapsto$	y.

So the resulting self-projectivity of ℓ fixes f_1 and f_2, and takes x to y. □

9.3 Group-theoretic interpretation of the Pappus property

By Proposition 9.2.6, $\text{Proj}(\ell)$ is always 3-transitive on ℓ. Sharpness of the action is equivalent to the Pappus property.

9.3.1. Proposition. *Let $\mathbb{P}$ be a projective plane. Then* $\text{Proj}(\ell)$ *is sharply 3-transitive on ℓ iff $\mathbb{P}$ is pappian.*

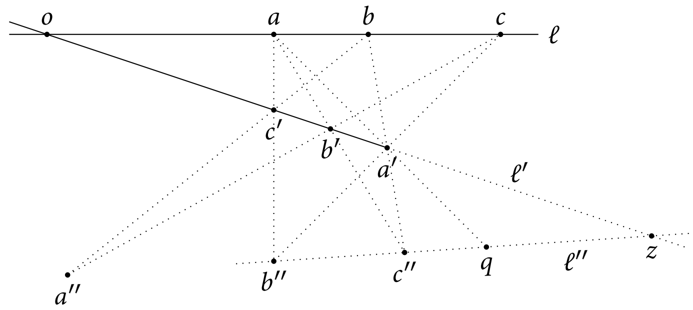
Proof. Suppose $\mathbb{P}$ is pappian. By Corollary 6.3.2, there is a commutative field $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$. Now the action of $\text{Proj}(\ell)$ on ℓ is equivalent to that of $\text{PGL}_2(\mathbb{F})$ on $\mathbb{F} \cup \{\infty\}$, which is known to be sharply 3-transitive (exercise 8.4.1).

For the converse we cannot use exercise 8.4.1, because we do not have a coordinate system yet; one must give a purely geometric argument. Let (o, a, b, c, a', b', c') be a Pappus configuration. Let $a'' = (bc') \cap (cb')$, $b'' = (ac') \cap (ca')$, and $c'' = (ab') \cap (ba')$. We want to show that a'', b'', c'' are collinear. Set $\ell'' = (b''c'')$; we shall prove $a'' \in \ell''$.

We introduce two auxiliary points:

- $q = (aa') \cap \ell''$;

- $z = \ell' \cap \ell''$.



We must prove $a'' \in (b''c'')$.

First consider the following projectivity and its effect:

ℓ	$\underset{a'}{\longrightarrow}$	ℓ''	$\underset{a}{\longrightarrow}$	ℓ'
a	$\mapsto$	q	$\mapsto$	a'
b	$\mapsto$	c''	$\mapsto$	b'
c	$\mapsto$	b''	$\mapsto$	c'
o	$\mapsto$	z	$\mapsto$	z

We need two more auxiliary points, which we do not show on picture:

- $\alpha'' = (b'c) \cap \ell''$;
- $\gamma' = (b\alpha'') \cap \ell'$.

Next consider this other projectivity (undefined point x plays no real role, since $(xb) = (bb')$):

ℓ	$\underset{b'}{\longrightarrow}$	ℓ''	$\underset{b}{\longrightarrow}$	ℓ'
a	$\mapsto$	c''	$\mapsto$	a'
b	$\mapsto$	x	$\mapsto$	b'
c	$\mapsto$	α''	$\mapsto$	γ'
o	$\mapsto$	z	$\mapsto$	z

Now the assumption that $\mathrm{Proj}(\ell)$ is sharply 3-transitive on ℓ immediately implies that for given lines m, m', there is a unique projectivity $m \to m'$ taking a distinct triple of m to one of m'. This means that $\gamma' = c'$.

Hence $\alpha'' \in (b\gamma') = (bc')$ and $\alpha'' \in (b'c)$ so $\alpha'' = a''$ lies on $\ell'' = (b''c'')$, as desired. □

9.4 Exercises

9.4.1. Exercise. *Return to the proof of Step 2 in Proposition 9.1.2. Which Desargues property is used?*

9.4.2. Exercise (cross-ratio). *In this exercise,* $\mathbb{F}$ *is a* commutative *field. We consider the usual action of* $\mathrm{PGL}_2(\mathbb{F})$ *on* $\ell = \mathbb{P}^1(\mathbb{F})$. *We extend field arithmetic to include* ∞ *with* $1/0 = \infty$, *and so on. The* cross-ratio *of* x_1, x_2, x_3, x_4 *is:*

$$\gamma(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}.$$

Prove that the group of bijections of ℓ preserving γ is exactly $\mathrm{PGL}_2(\mathbb{F})$.

9.4.3. Exercise. *Give a geometric proof that if $\mathbb{P}$ is pappian, then* $\mathrm{Proj}(\ell)$ *is sharply* 3-*transitive. (First prove that any product of self-projectivities is a product of at most two perspectivities.)*

10 Artin's coordinatisation (1): dilations and translations

Abstract. This section and the next give a more conceptual proof of Hilbert coordinatisation using group theory. § 10.1 introduces certain automorphisms called *dilations*. § 10.2 studies those without a fixed point, called *translations*. § 10.3 investigates them further, assuming a weak form of the Desargues property.

Throughout $\mathbb{A}$ is an affine plane.

- We do *not* always assume desarguesianity. Instead, the results will describe the progressive effects of increasingly stronger forms of the Desargues property.
- It will be convenient, when working with automorphisms $f\colon \mathbb{A} \to \mathbb{A}$, to use classical 'high-school' notation $x' = f(x)$.

10.1 Dilations

The group of automorphisms of $\mathbb{A}^2(\mathbb{F})$ is $\mathbb{F}^2_+ \rtimes \Gamma\Lambda(\mathbb{F}^2)$. To 'reconstruct $\mathbb{F}$ from $\mathbb{A}^2(\mathbb{F})$', one may prefer a soluble subgroup. On the other hand, behaviour of $\mathrm{Aut}(\mathbb{A})$ in an abstract affine plane $\mathbb{A}$ is quite unpredictable. Artin's method is to focus on a special subgroup of $\mathrm{Aut}(\mathbb{A})$.

10.1.1. Definition (the group of dilations)**.** A *dilation* of $\mathbb{A}$ is an automorphism which takes any line to a parallel line, viz. satisfying:

$$(\forall a, b)[a \neq b \to (a'b') \mathbin{/\!\!/} (ab)].$$

Let $\mathrm{Dil}(\mathbb{A})$ be the group they form under composition.

10.1.2. Remarks.

- In projective terms, a dilation is an automorphism of $\hat{\mathbb{A}}$ fixing the line at infinity pointwise (more precisely: the restriction to $\mathbb{A}$ of such an automorphism):

$$\mathrm{Dil}(\mathbb{A}) = \mathrm{Stab}_{\mathrm{Aut}(\hat{\mathbb{A}})}(\ell_\infty).$$

- In $\mathbb{A}^2(\mathbb{F})$, one has $\mathrm{Dil}(\mathbb{A}) \simeq \mathbb{F}^2_+ \rtimes \mathbb{F}^\times$.

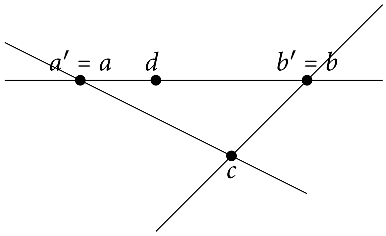
10.1.3. Proposition.

(i) If a dilation f has two distinct fixed points, then $f = \mathrm{Id}_{\mathbb{A}}$.

(ii) Any dilation is determined by the image of two distinct points.

Proof.

(i) Suppose f fixes a and b. Using our convention to write x' for $f(x)$, this means $a' = a$ and $b' = b$. The proof will introduce two cases, points *not* on (ab) being easier to understand. This kind of case division will be common.



- Let $c \notin (ab)$. Then $a = a' \in (a'c') /\!/ (ac)$ since f is a dilation; so by uniqueness in $\mathbf{AP_2}$ one has $(ac') = (ac)$ and $c' \in (ac)$. Likewise $c' \in (bc)$; these lines meet at c, so $c' = c$ is fixed. This proves that any point *not* on (ab) is fixed.
- Now let $d \in (ab)$. Using any $c \notin (ab)$ (there exists such a point by $\mathbf{AP_3}$), one has $c' = c$ and $d \notin (ac)$. By the previous argument, d is fixed too.

Thus all points are fixed and $f = \mathrm{Id}_{\mathbb{A}}$.

(ii) If $f_1, f_2 \in \mathrm{Dil}(\mathbb{A})$ agree on two different points $a \neq b$, then $f_1^{-1} f_2 \in \mathrm{Dil}(\mathbb{A})$ fixes a and b, so by (i), one has $f_1 = f_2$. □

In $\mathbb{A}^2(\mathbb{F})$, dilations are either translations or homotheties. The difference is in the number of fixed points. The next subsections build on this observation, in an abstract affine plane.

10.2 Translations

10.2.1. Definition (translation)**.** A *translation* of $\mathbb{A}$ is either the identity map $\mathrm{Id}_{\mathbb{A}}$, or a fixed point-free dilation. Let $\mathrm{Trans}(\mathbb{A})$ be their *set* (one must prove that it is a group).

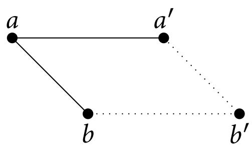
10.2.2. Remark. The terminology is consistent with ordinary practice: if $\mathbb{A} = \mathbb{A}^2(\mathbb{F})$, then a translation in this sense is one in the usual sense.

Remember that given a point a and line ℓ, we use ℓ_a to denote the unique line parallel to ℓ through a. Moreover, for t a translation $\neq \mathrm{Id}_{\mathbb{A}}$ and a a point, one has $t(a) \neq a$ so line $(at(a))$ is well-defined.

10.2.3. Proposition.

(i) If t is a translation $\neq \mathrm{Id}_{\mathbb{A}}$ then for any $a, b \in \mathbb{A}$, one has $(at(a)) /\!/ (bt(b))$.

(ii) If $t \neq \mathrm{Id}_{\mathbb{A}}$ is a translation and $b \notin (aa')$ then $b' = (ab)_{a'} \cap (aa')_b$.



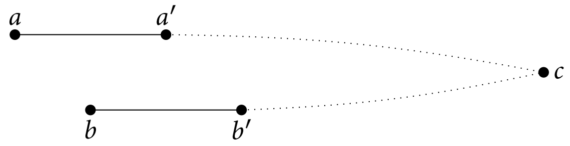
(iii) $\mathrm{Trans}(\mathbb{A}) \trianglelefteq \mathrm{Dil}(\mathbb{A})$ *is a normal subgroup.*

(iv) *If two translations coincide at one point, they are equal.*

No claims on abelianity so far.

Proof. Whenever working with only one dilation (eg. a translation), we implicitly use notation $x' = f(x)$.

(i) Suppose there is a counterexample. Then lines (aa') and (bb'); let $c = (aa') \cap (bb')$.

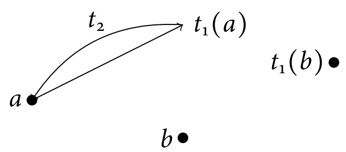


Now $c' \in (a'c') /\!\!/ (ac)$ since t is a dilation. But a, a', c are collinear so $c' \in (aa')$. Likewise, $c' \in (bb')$ and $c' = c$. Thus t has a fixed point: a contradiction.

(ii) Since t is a dilation, $(a'b') /\!\!/ (ab)$. But also, since t is a translation, (i) implies that $(bb') /\!\!/ (aa')$. Now $b \notin (aa')$, so lines (aa') and (ab) are not parallel. Therefore lines $(a'b')$ and (bb') are not parallel, and their intersection is the point b'.

(iii) By definition, $\mathrm{Id}_{\mathbb{A}}$ is a translation; clearly the inverse of a translation is a translation; moreover if $(t, d) \in \mathrm{Trans}(\mathbb{A}) \times \mathrm{Dil}(\mathbb{A})$ then dtd^{-1} has no fixed point. So $\mathrm{Trans}(\mathbb{A})$ is a normal *subset* of $\mathrm{Dil}(\mathbb{A})$. It only remains to prove that $\mathrm{Trans}(\mathbb{A})$ is stable under composition.

Let t_1, t_2 be translations; we show $t_2 t_1 \in \mathrm{Trans}(\mathbb{A})$. We may supppose that neither of t_1, t_2 is $\mathrm{Id}_{\mathbb{A}}$. To prove that the dilation $t_2 t_1$ is a translation, we show that it is either fixed point-free or $\mathrm{Id}_{\mathbb{A}}$. So suppose $t_2 t_1$ has a fixed point a.



Pick any $b \notin (a\ t_1(a))$. We localise $t_2 t_1(b)$ on two non-parallel lines.

- On the one hand:

$$
\begin{aligned}
((t_2(b)\ t_2 t_1(b)) &/\!\!/ (b\ t_1(b)) && \text{since } t_2 \in \mathrm{Dil}(\mathbb{A})\\
&/\!\!/ (a\ t_1(a)) && \text{since } t_1 \in \mathrm{Trans}(\mathbb{A}), \text{by (i)}\\
&/\!\!/ (t_2(a)\ t_2 t_1(a)) && \text{since } t_2 \in \mathrm{Dil}(\mathbb{A})\\
&= (a\ t_2(a)) && \text{since } t_2 t_1 \text{ fixes } a\\
&/\!\!/ (b\ t_2(b)) && \text{since } t_2 \in \mathrm{Trans}(\mathbb{A}), \text{by (i).}
\end{aligned}
$$

It follows that $t_2 t_1(b) \in (b\ t_2(b))$.

- On the other hand, because $t_2 t_1$ is a dilation fixing a:

$$(a\ t_2 t_1(b)) = (t_2 t_1(a)\ t_2 t_1(b)) \mathbin{/\!/} (ab),$$

so $t_2 t_1(b) \in (ab)$.

If lines $(b\ t_2(b))$ and (ab) are parallel, then reading the above computations we see that $(ab) = (a\ t_1(a))$, against the choice of b. So lines $(b\ t_2(b))$ and (ab) are *not* parallel, and their intersection is exactly b. Therefore $t_2 t_1(b) = b$. But a dilation fixing two points is the identity by Proposition 10.1.3. We are done.

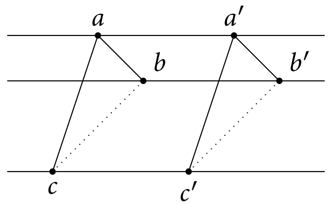
(iv) If $t_1(a) = t_2(a)$, then $t' = t_2^{-1} t_1$ is a translation by (iii). It has a fixed point, so $t_2^{-1} t_1 = \mathrm{Id}_{\mathbb{A}}$. □

10.3 The (p,p)-Desargues property

The continued study of the group of translations needs a (weak) form of desarguesianity of $\mathbb{A}$.

10.3.1. Definition ((p,p)-Desargues)**.** The *(parallel, parallel)-form of the Desargues property* (for short: *(p, p)-Desargues*) is the following axiom.

> Let $(aa'), (bb'), (cc')$ be three distinct parallel lines. Suppose $(ab) \mathbin{/\!/} (a'b')$ and $(ac) \mathbin{/\!/} (a'c')$. Then $(bc) \mathbin{/\!/} (b'c')$.



Two triangles on three parallel lines. If two pairs of sides are parallel, so is the third.

The property is also called the *small Desargues axiom*.

10.3.2. Proposition.

(i) $\mathbb{A}$ *has (p,p)-Desargues iff* $\mathrm{Trans}(\mathbb{A})$ *is transitive on* $\mathbb{A}$.

(ii) If this holds, then $\mathrm{Trans}(\mathbb{A})$ *is abelian.*

10.3.3. Remarks.

- Since distinct translations never coincide anywhere, there can be at most one translation t mapping given a to given b. So in (i), transitivity is equivalent to sharp transitivity.
- The converse of (ii) need not hold: one can construct affine planes with $\mathrm{Trans}(\mathbb{A}) = \{\mathrm{Id}\}$ (which is abelian, but not transitive).

Proof.

(i) First suppose that $\mathrm{Trans}(\mathbb{A})$ is transitive on $\mathbb{A}$; we prove (p,p)-Desargues. Take a (p,p)-Desargues configuration $(a, b, c; a', b', c')$; suppose $(ab) /\!/ (a'b')$ and $(ac) /\!/ (a'c')$. We want to show $(bc) /\!/ (b'c')$. By transitivity, let t be the translation mapping a to a'. Then by Proposition (ii), one has $t(b) = (ab)_{a'} \cap (aa')_b = b'$ and $t(c) = c'$ likewise. Now since t is a dilation, $(b'c') /\!/ (bc)$, as desired.

The converse is more interesting, and requires *partial* maps. Suppose (p,p)-Desargues holds. For any pair of distinct points (a, a'), we define a partial map $\check{t}_{a,a'} : \mathbb{A} \setminus (aa') \to \mathbb{A}$ by:

$$\check{t}_{a,a'}(b) = (ab)_{a'} \cap (aa')_b.$$

This is well-defined, but a partial map.

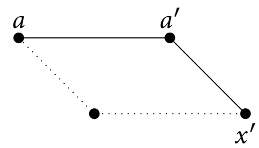
Step 1. If $b \notin (aa')$ and $b' = \check{t}_{a,a'}(b)$, then $\check{t}_{a,a'}$ and $\check{t}_{b,b'}$ agree wherever both are defined.

Verification. This is exactly the (p,p)-Desargues assumption. ◇

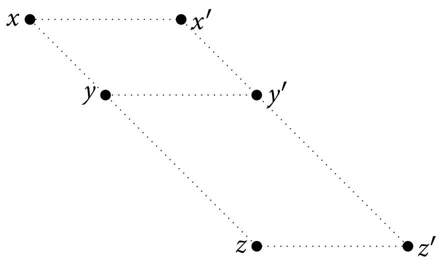
This defines a global map $t_{a,a'} : \mathbb{A} \to \mathbb{A}$, and $t_{a,a'}(a) = a'$. We must prove that $t_{a,a'}$ is a translation. Notice that if $t_{a,a'}(b) = b'$, then $t_{a,a'} = t_{b,b'}$.

Step 2. $t_{a,a'}$ is a dilation.

Verification. Bijectivity is obvious from the following picture.



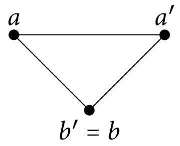
We claim that $t_{a,a'}$ is an automorphism. Suppose x, y, z are collinear (and we may suppose they are distinct); we show that so are the images x', y', z'.



But we know that $t_{a,a'} = t_{x,x'}$, so actually $y' = (xx')_y \cap (xy)_{x'} \in (xy)_{x'}$ and $z' \in (xz)_{x'} = (xy)_{x'}$ by collinearity of x, y, z. This proves collinearity of x', y', z'.

We claim that t is a dilation. But we already know that $(x'y') /\!/ (xy)$, so this is clear. ◇

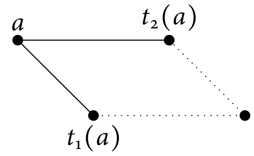
Finally t is a translation. Otherwise it has a fixed point b.



Up to changing base point we may suppose $b \notin (aa')$. Now $(ab) /\!\!/ (a'b') = (a'b)$ so $b \in (aa')$, a contradiction.

(ii) Let t_1, t_2 be translations. We want to show $t_2 t_1 = t_1 t_2$; we may assume that neither is $\mathrm{Id}_{\mathbb{A}}$, so neither has a fixed point. Let $a \in \mathbb{A}$.

Case 1. Suppose $a, t_1(a), t_2(a)$ are *not* collinear.



Then by Proposition (ii):

$$t_2(t_1(a)) = (a\, t_1(a))_{t_2(a)} \cap (a\, t_2(a))_{t_1(a)} = t_1 t_2(a),$$

as desired.

Case 2. Suppose $a, t_1(a), t_2(a)$ *are* collinear, say on ℓ. The above no longer applies. By $\mathbf{AP_3}$ take $b \notin \ell$. By transitivity, there is a translation t_0 taking a to b. If $t_1 t_0(a) = t_1(b) \in \ell$, then $\ell = (t_1(a)\, t_1(b)) /\!\!/ (ab)$ and $b \in (ab) = \ell$, a contradiction. So $t_1 t_0(a) \notin \ell$. Therefore, by case 1 applied to pairs $(t_1 t_0, t_2)$ and (t_0, t_2):

$$\begin{aligned} t_2 t_1 &= (t_2 t_1)(t_0 t_0^{-1}) = t_2(t_1 t_0) t_0^{-1} \\ &= (t_1 t_0)(t_2 t_0^{-1}) = t_1 (t_0 t_2) t_0^{-1} \\ &= t_1 t_2 t_0 t_0^{-1} = t_1 t_2, \end{aligned}$$

and we are done again. □

10.3.4. Remark. Abelianity can be proved assuming only the existence of translations having different 'directions' (see Definition 11.2.2 below). The latter is proved to be a consequence of (p, p)-Desargues, but is weaker.

10.4 Exercises

10.4.1. Exercise. *Let $f \neq \mathrm{Id}_{\mathbb{A}}$ be a dilation. Suppose that for any $a, b \in \mathbb{A}$ one has $(aa') /\!\!/ (bb')$. Prove that f is a translation.*

11 Artin's coordinatisation (2): the skew-field

Abstract. A sequel to § 10. We recover a vector geometry from an affine plane, paying attention to assumptions. § 11.1 introduces *homotheties* and another form of the Desargues property. § 11.2 constructs the ring of *direction-preserving* endomorphisms of Trans($\mathbb{A}$), provided the latter is abelian. § 11.3 shows it is a skew-field, provided $\mathbb{A}$ has (p,p)-Desargues. § 11.4 finishes coordinatisation, provided $\mathbb{A}$ has (c, p)-Desargues.

11.1 Homotheties and the (c,p)-Desargues property

We construct and interpret homotheties by arguments similar to translations.

11.1.1. Definition (homothety)**.** A *homothety* of $\mathbb{A}$ is a dilation having at least one fixed point (this includes $\mathrm{Id}_{\mathbb{A}}$).

For $a \in \mathbb{A}$, let $\mathrm{Hom}_a(\mathbb{A})$ be the group of dilations fixing a.

There is no equivalent of Proposition 10.2.3. Two homotheties with the same fixed point which coincide at one other point, are equal; but this follows from Proposition 10.1.3.

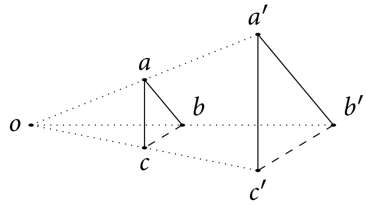
11.1.2. Corollary. *If* $\mathbb{A}$ *has (p, p)-Desargues then* $\mathrm{Dil}(\mathbb{A}) = \mathrm{Trans}(\mathbb{A}) \rtimes \mathrm{Hom}_a(\mathbb{A})$ *for any* $a \in \mathbb{A}$.

Proof. By Proposition (iii), $\mathrm{Trans}(\mathbb{A}) \trianglelefteq \mathrm{Dil}(\mathbb{A})$; by definition, $\mathrm{Trans}(\mathbb{A}) \cap \mathrm{Hom}_a(\mathbb{A}) = \{\mathrm{Id}_{\mathbb{A}}\}$. Now let $d \in \mathrm{Dil}(\mathbb{A})$ be any dilation. By (p, p)-Desargues there is a translation t with $t(a) = d(a)$. Hence $t^{-1}d(a) = a$ so that $t^{-1}d \in \mathrm{Hom}_a(\mathbb{A})$. Therefore $d \in \mathrm{Trans}(\mathbb{A}) \rtimes \mathrm{Hom}_a(\mathbb{A})$. ☐

We give the relevant form of the Desargues property.

11.1.3. Definition ((c,p)-Desargues)**.** The *(concurrent, parallel)-form of the Desargues property* (for short: *(c, p)-Desargues*) is the following axiom.

> Let $(aa'), (bb'), (cc')$ be three distinct concurrent lines. Suppose $(ab) /\!/ (a'b')$ and $(ac) /\!/ (a'c')$. Then $(bc) /\!/ (b'c')$.



Two triangles on three concurrent lines. If two pairs of sides are parallel, so is the third.

The property is also called the *big Desargues axiom*.

11.1.4. Proposition. $\mathbb{A}$ *has (c,p)-Desargues iff for any line* ℓ *and point* $o \in \ell$, $\mathrm{Hom}_o(\mathbb{A})$ *acts transitively on* $\ell \setminus \{o\}$.

11.1.5. Remark. Here again, since an element of $\mathrm{Hom}_o(\mathbb{A})$ already fixes o and a non-identity dilation has at most one fixed point, sharpness is for free.

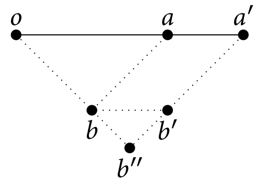
Proof. Exercise 11.5.1. □

11.1.6. Corollary. *If $\mathbb{A}$ has (c, p)-Desargues, then it has (p, p)-Desargues.*

11.1.7. Remark. This corollary explains the classical (rather obscure) terminology 'big Desargues axiom' versus 'small Desargues axiom': one is stronger than the other.

Proof. The statement is not obvious geometrically. We give an algebraic proof, using the equivalences of Propositions 10.3.2 and 11.1.4. It suffices to show that $\mathrm{Trans}(\mathbb{A})$ acts transitively. Let a, a' be distinct points of $\mathbb{A}$.

Let $o \in (aa') \setminus \{a, a'\}$ (here we must assume that lines have more than three points). By assumption, there is $g_1 \in \mathrm{Hom}_o(\mathbb{A})$ such that $g_1(a) = a'$. Also let $b \notin (ao)$, and let $b' = (ab)_{a'} \cap (aa')_b$, then $b'' = g_1(b)$. By assumption, there is $g_2 \in \mathrm{Hom}_{a'}(\mathbb{A})$ such that $g_2(b'') = b'$.



We claim that $t = g_2 g_1$ is a translation taking a to a'. Of course $t(a) = g_2(a') = a'$, but also $t(b) = g_2(b'') = b'$. So if t has a fixed point c, then one must have $(ac) \parallel (a'c') = (a'c)$, and therefore $c \in (aa')$. But $c \in (bb')$ likewise, and these lines do *not* meet. Hence t has no fixed point; it is a translation. □

11.2 The ring of direction-preserving translations

Here is another idea. In coordinatised affine planes, one has $\mathrm{Trans}(\mathbb{A}^2(\mathbb{F})) \simeq \mathbb{F}^2$, which is a vector space over $\mathbb{F}$. But a priori, $\mathrm{Trans}(\mathbb{A}^2(\mathbb{F}))$ is merely a group. One may retrieve $\mathbb{F}$ as a special ring of endomorphisms of $\mathrm{Trans}(\mathbb{A}^2(\mathbb{F}))$; the abstract formulation requires two definitions.

11.2.1. Definition (ring of endomorphisms of an abelian group)**.** Let $(M;+)$ be an *abelian* group. Then the set $\mathrm{End}(M) = \{f\colon (M;+) \to (M;+)\}$ of *endomorphisms* of the group $(M;+)$ forms a ring under:

- pointwise sum, viz. $(f+g)(x) = f(x) + g(x)$;
- composition, viz. $(g \cdot f)(x) = g(f(x))$.

The respective identity elements are the zero (constant) map 0_M and the identity map Id_M.

In general, the ring is non-commutative.

11.2.2. Definition (direction of a translation)**.**

- The *direction* of a translation $t \neq \mathrm{Id}_{\mathbb{A}}$ is the equivalence class of $(a\ t(a))$ for any $a \in \mathbb{A}$. (This is well-defined by Proposition (i).) We denote it by d_t.

- An endomorphism $\lambda\colon \mathrm{Trans}(\mathbb{A}) \to \mathrm{Trans}(\mathbb{A})$ is *direction-preserving* if: for any $t \in \mathrm{Trans}(\mathbb{A}) \smallsetminus \{\mathrm{Id}_{\mathbb{A}}\}$, one has $\lambda(t) \neq \mathrm{Id}_{\mathbb{A}}$ and $d_{\lambda(t)} = d_t$.

Let $\mathbb{F} = \{\lambda \in \mathrm{End}(\mathrm{Trans}(\mathbb{A})) : \lambda \text{ is direction-preserving}\} \cup \{0\}$.

In particular a direction-preserving endomorphism of $\mathrm{Trans}(\mathbb{A})$ is injective by construction.

11.2.3. Proposition. *Let $\mathbb{A}$ be an affine plane such that* $\mathrm{Trans}(\mathbb{A})$ *is abelian. Then $\mathbb{F}$ is a subring of* $\mathrm{End}(\mathrm{Trans}(\mathbb{A}))$.

Proof. Clearly $\mathbb{F}$ contains 0 and 1; so it suffices to check stability under $+, -, \cdot$. Opposition will be an exercise.

Step 1. $\mathbb{F}$ is stable under $+$.

Verification. Let $\lambda, \mu \in \mathbb{F}$; we show $\lambda + \mu \in \mathbb{F}$. We may assume that neither λ, μ, nor $\lambda + \mu$ is 0. Let $t \in \mathrm{Trans}(\mathbb{A}) \smallsetminus \{\mathrm{Id}_{\mathbb{A}}\}$. By definition, λ is injective, so $\lambda(t) \neq \mathrm{Id}_{\mathbb{A}}$ and $\mu(t) \neq \mathrm{Id}_{\mathbb{A}}$ likewise. Also $\lambda(t) + \mu(t) \neq \mathrm{Id}_{\mathbb{A}}$ as otherwise $\lambda + \mu = 0$.

Since λ et μ are direction-preserving, $d_{\lambda(t)} = d_t = d_{\mu(t)}$. Now this is also the direction of $\lambda(t) + \mu(t)$, so $d_t = d_{\lambda(t)+\mu(t)} = d_{(\lambda+\mu)(t)}$, so $\lambda + \mu$ is direction-preserving. ◇

Step 2. $\mathbb{F}$ is stable under $\cdot$.

Verification. Let $\lambda, \mu \in \mathbb{F}$; we may assume that neither λ nor μ is 0. Then both are injective, and therefore $\lambda \cdot \mu \neq 0$. Now for any $t \in \mathrm{Trans}(\mathbb{A}) \smallsetminus \{\mathrm{Id}_{\mathbb{A}}\}$, $d_{(\lambda\cdot\mu(t)} = d_{\lambda(\mu(t))} = d_{\mu(t)} = d_t$, so by definition, $\lambda \cdot \mu \in \mathbb{F}$. ◇

Notice that since $\mathrm{End}(\mathrm{Trans}(\mathbb{A}))$ is a ring, $\mathbb{F}$ inherits associativity and distributivity without a need to check them. □

11.3 A skew-field assuming (p,p)-Desargues

Proposition 11.2.3 does not use any form of Desargues property, only abelianity of $\mathrm{Trans}(\mathbb{A})$.

11.3.1. Definition. For $h \in \mathrm{Dil}(\mathbb{A})$ any dilation, consider the transformation:

$$\begin{array}{rccc} \lambda_h\colon & T & \to & T \\ & t & \mapsto & hth^{-1}. \end{array}$$

Notice that λ_h is always a group automorphism of $\mathrm{Trans}(\mathbb{A})$, and direction-preserving (by definition of a dilation): hence $\lambda_h \in \mathbb{F}$.

Recall from Proposition 10.3.2 that (p, p)-Desargues implies abelianity of $\mathrm{Trans}(\mathbb{A})$.

11.3.2. Proposition. *If $\mathbb{A}$ has (p, p)-Desargues, then $\mathbb{F}$ is a skew-field. Moreover, for any point a one has* $\mathrm{Hom}_a(\mathbb{A}) \simeq \mathbb{F}^\times$.

Proof. The proof entirely relies on a geometric lemma, proved *after* its consequences.

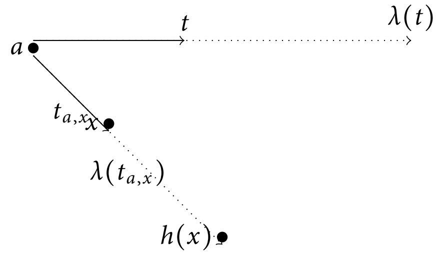
11.3.3. Lemma. *Suppose that $\mathbb{A}$ has (p,p)-Desargues. Let $a \in \mathbb{A}$ be fixed. Then for all*

$\lambda \in \mathbb{F} \setminus \{0\}$ *there is a unique* $h \in \mathrm{Hom}_a(\mathbb{A})$ *with* $\lambda_h = \lambda$.

We first show how it implies the Proposition.

- Let $\lambda \in \mathbb{F} \setminus \{0\}$. By the Lemma, $\lambda = \lambda_h$ for some h. Let $\mu = \lambda_{h^{-1}}$. Then $\mu \in \mathbb{F}$ and $\lambda\mu = \mu\lambda = 1$ in $\mathbb{F}$, so $\mathbb{F}$ is a skew-field.
- There is a group homomorphism $\mathrm{Hom}_a(\mathbb{A}) \twoheadrightarrow \mathbb{F}^\times$; this is actually an isomorphism. For if $h \in \mathrm{Hom}_a(\mathbb{A})$ lies in the kernel, then it means that $\lambda_h = 1$, or equivalently, that for any $t \in \mathrm{Trans}(\mathbb{A})$, one has $ht = th$. However as soon as $t \neq \mathrm{Id}_{\mathbb{A}}$, one gets $ht(a) = th(a) = t(a)$ so h fixes not only a but also $t(a) \neq a$; being a dilation it implies $h = \mathrm{Id}_{\mathbb{A}}$, as desired.

Proof of the Lemma.



By (p,p)-Desargues, for any $x \in \mathbb{A}$ there is a translation $t_{a,x}$ taking a to x. Define:

$$h(x) = (\lambda(t_{a,x}))\,(a)$$

We claim that $h \in \mathrm{Hom}_a(\mathbb{A})$ and $\lambda_h = \lambda$.

Step 1. $h \in \mathrm{Hom}_a(\mathbb{A})$.

Verification. First notice that h is well-defined since $t_{a,x}$ exists and is unique. Also $h(a) = \lambda(\mathrm{Id}_{\mathbb{A}})(a) = \mathrm{Id}_{\mathbb{A}}(a) = a$ since λ is a group endomorphism. But we have a number of things to check that it is a dilation.

Let $y \in \mathbb{A}$ be another point. Then $t_{a,y} = t_{x,y}t_{a,x}$ since these two translations coincide at a. Therefore, using that λ is a group endomorphism:

$$h(y) = \big(\lambda(t_{x,y})\lambda(t_{a,x})\big)\,(a) = \lambda(t_{x,y})(h(x))$$

In particular, if $x \neq y$, then $t_{x,y} \neq \mathrm{Id}_{\mathbb{A}}$ and since $\lambda \in \mathbb{F} \setminus \{0\}$ is direction-preserving, $\lambda(t_{x,y}) \neq \mathrm{Id}_{\mathbb{A}}$, proving $h(y) \neq h(x)$. So h is injective. But also, always since λ is direction-preserving,

$$(h(x)\,h(y)) \,/\!\!/\, d_{t_{x,y}} \,/\!\!/\, (xy)$$

An injective map with this property is always a dilation (preservation of collinearity and surjectivity being consequences left as an exercise). So $h \in \mathrm{Dil}(\mathbb{A})$ and since we already know $h(a) = a$, we have $h \in \mathrm{Hom}_a(\mathbb{A})$. ◇

Step 2. $\lambda_h = \lambda$.

Verification. Take any $t \in \mathrm{Trans}(\mathbb{A})$; say $t = t_{a,b}$ for some b. Then $\lambda_h(t) = hth^{-1}$ is the translation taking a to $hth^{-1}(a) = ht(a) = h(b) = \lambda(t_{a,b})(a) = \lambda(t)(a)$. So $\lambda_h(t) = \lambda(t)$, for any translation; hence $\lambda_h = \lambda$ in $\mathbb{F}$. ◇

So $h \in \mathrm{Hom}_a(\mathbb{A})$ and $\lambda_h = \lambda$. □

This proves the proposition. □

11.4 Coordinatisation assuming (c, p)-Desargues

One can thus retrieve a skew-field under as weak a property as (p,p)-Desargues. Now the $\mathbb{F}$-module $\mathrm{Trans}(\mathbb{A})$ is actually a vector space. One may think this is close to coordinatising, yet the dimension theory of $\mathrm{Trans}(\mathbb{A})$ is not clear at all.

Recall that the direction (Definition 11.2.2) of a translation t is denoted by d_t.

11.4.1. Proposition. *Suppose $\mathbb{A}$ has (c, p)-Desargues. Then:*

(i) for $t, t' \in \mathrm{Trans}(\mathbb{A}) \setminus \{\mathrm{Id}\}$, one has $d_t = d_{t'}$ iff there is $\lambda \in \mathbb{F}$ with $t' = \lambda t$;

(ii) $\mathrm{Trans}(\mathbb{A})$ is 2-dimensional as a left-vector space over $\mathbb{F}$;

(iii) $\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$.

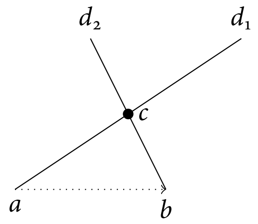
Proof. Recall from Corollary 11.1.6 that (c, p)-Desargues implies (p, p)-Desargues, which in turn implies (sharp) transitivity of $\mathrm{Trans}(\mathbb{A})$ on $\mathbb{A}$ by Proposition 10.3.2. Whenever $a \neq b \in \mathbb{A}$, we let $t_{a,b}$ be the unique translation taking a to b. Its direction is then the parallelism class of (ab).

(i) Let $t, t' \in \mathrm{Trans}(\mathbb{A}) \setminus \{\mathrm{Id}_{\mathbb{A}}\}$. If there is $\lambda \in \mathbb{F}$ with $t' = \lambda t$ then $\lambda \neq 0$, so by definition, λ is direction-preserving; hence $d_{t'} = d_t$.

Now suppose $d_{t'} = d_t$, and say $t = t_{o,a}$, $t' = t_{o,a'}$. By (c, p)-Desargues, there is $h \in \mathrm{Hom}_o(\mathbb{A})$ doing $h(a) = a'$. Then let $\lambda = \lambda_h$. We know that:

$$\begin{aligned}\lambda t &= hth^{-1}\\ &= t_{o,hth^{-1}(o)}\\ &= t_{o,ht(o)}\\ &= t_{o,h(a)}\\ &= t_{o,a'}\\ &= t'.\end{aligned}$$

(ii) It suffices to show that two translations with different directions span $\mathrm{Trans}(\mathbb{A})$. These exist under (p, p)-Desargues. Let t_1, t_2 have directions $d_1 \neq d_2$. Let t be another translation. By (i) we may suppose that the direction of t is neither d_1 nor d_2. Fix $a \in \mathbb{A}$ and let $b = t(a)$.



Lines $(d_1)_b$ and $(d_2)_a$ are not parallel, so they meet at say c. Clearly c is neither a nor b. So $t_{c,b}$ has direction d_1, which is the same as t_1. By (i), there is $\lambda \in \mathbb{F}$ with $t_{c,b} = \lambda t_1$. Likewise, the direction of $t_{a,c}$ is d_2, so there is $\mu \in \mathbb{F}$ with $t_{a,c} = \mu t_2$. Now:

$$t = t_{a,b} = t_{c,b} \circ t_{a,c} = (\lambda(t_1)) \circ (\mu(t_2)),$$

as transformations of $\mathbb{A}$, which in vector space notation rewrites: $t = \lambda \cdot t_1 + \mu \cdot t_2$.

(iii) Fix any $o \in \mathbb{A}$. Then by sharp transitivity, $\mathbb{A} = \mathrm{Trans}(\mathbb{A}) \cdot o$ is parametrised by $\mathrm{Trans}(\mathbb{A}) \simeq \mathbb{F}^2$ as a left-vector space over $\mathbb{F}$. So we may represent points in $\mathbb{A}$ by pairs in $\mathbb{F}^2$.

Let $p_1 \neq p_2$ be two points in $\mathbb{A}$. Then for any $q \in \mathbb{A}$ one has equivalences:

$$\begin{aligned} q \in (p_1 p_2) &\text{ iff } q = p_1 \text{ or } (p_1 q) \parallel (p_1 p_2) \\ &\text{ iff } q = p_1 \text{ or } d_{t_{p_1 q}} = d_{t_{p_1 p_2}} \\ &\text{ iff } (\exists \lambda \in \mathbb{F})(t_{p_1,q} = \lambda t_{p_1,p_2}). \end{aligned}$$

So $(p_1 p_2) = \{p_1 + \lambda \cdot t_{p_1,p_2} : \lambda \in \mathbb{F}\}$. Thus lines in $\mathbb{A}$ correspond to parametric affine lines in $\mathbb{F}^2$. This gives an isomorphism $\mathbb{A} \simeq \mathbb{A}^2(\mathbb{F})$. □

11.4.2. Remarks.

- I do not know which finite values of $\dim_{\mathbb{F}} \mathrm{Trans}(\mathbb{A})$ are possible in abstract affine planes.
- Objects $\mathrm{Trans}(\mathbb{A})$ and $\mathbb{F}$ are *intrinsic*, viz. do not depend on anything. On the other hand coordinatisation does: it depends on the choice of o. Notice that we do *not* introduce other extrinsic data; Artin' method requires no axes.
- Hilbert's and Artin's methods have in common that one has to go to the affine part of a desarguesian projective plane. I am not aware of proofs entirely taking place in the original projective plane.
- I am not aware either of quicker arguments directly coordinatising 'projective 3-spaces' (Definition 4.1.1).[19]

11.5 Exercises

11.5.1. Exercise. *Prove Proposition 11.1.4, viz.: $\mathbb{A}$ has (c,p)-Desargues iff for any line ℓ and point $o \in \ell$, $\mathrm{Hom}_o(\mathbb{A})$ acts transitively on $\ell \setminus \{o\}$.*

For the direct implication, first construct a partial map $\check{h}_{o,a\to a'}$ defined on $\mathbb{A} \setminus \ell$.

[19] To some extent, the following sketches an answer, but not a very satisfactory one as it resembles an intermediate stage between Hilbert and Artin. M. K. Bennett, Coordinatization of affine and projective space, *Discrete Mathematics* 4, pp. 219–231, 1973.

11.5.2. Exercise. *There is a (c, c)-Desargues property; draw its picture. Now give a* geometric *proof that (c, p)-Desargues implies (c, c)-Desargues.*[20]

(It is a consequence of coordinatisation that (c, p) implies all other affine forms.)

12 Duality

Abstract. § 12.1 dualises geometric statements; if a geometric statement φ holds in all projective planes, then so does its *dual statement* φ^*. § 12.2 dualises projective planes, viz. attaches to each projective plane a *dual plane*, with dual properties. § 12.3 studies the behaviour of the Desargues and Pappus properties under duality.

We use the following notation from mathematical logical. For Γ an incidence geometry and φ a geometric statement (one about points, lines, incidence), write $\Gamma \vDash \varphi$ if Γ satisfies φ.

12.1 Duality

Return to the definition of a projective plane (Definition 2.1.1). There is a noticeable symmetry in the axioms when one exchanges points and lines.

12.1.1. Definition (dual statement)**.** Let φ be a statement about projective planes. Its *dual statement* is the statement φ^* obtained by exchanging the words 'point' and 'line', and exchanging the symbols $\in$ and $\ni$.

12.1.2. Example. The dual statement of $\mathbf{PP_1}$ is $\mathbf{PP_1}^* = \mathbf{PP_2}$. Likewise, $\mathbf{PP_2}^* = \mathbf{PP_1}$. This is not surprising since $\varphi^{**} = \varphi$.

12.1.3. Theorem (duality theorem)**.** *Let φ be a statement about projective planes. Then φ holds in* every *projective plane iff φ^* does.*

Be careful that 'being desarguesian' does not hold in *every* projective plane, so it is not a suitable φ. But 'Pappus implies Desargues' is an example of such a statement.

Proof using mathematical logic. Since φ^{**} is always the formula φ, one implication is enough. We want to argue by symmetry of the axioms for projective planes. We know that $\mathbf{PP_1}^* = \mathbf{PP_2}$, but we should also do something about $\mathbf{PP_3}$.

Step 1. Let Γ be an incidence geometry satisfying $\mathbf{PP_1}$ and $\mathbf{PP_2}$. Then in Γ, $\mathbf{PP_3}$ is equivalent to:

$\mathbf{PP_3}^*$**.** There are four distinct lines, no three of which are concurrent.

Verification. Suppose $\mathbf{PP_3}$. There are four points a, b, c, d no three of which are collinear. So lines $(ab), (ac), (bc), (bd)$ are four in number and no three of them concur. Hence $\mathbf{PP_3}^*$ holds. The converse can be obtained similarly. ◇

Suppose that all projective planes satisfy φ. Then by 'Skolem-Gödel's completeness theorem' there is a proof of φ using only axioms $\mathbf{PP_1}, \mathbf{PP_2}, \mathbf{PP_3}$. Now exchange the

[20] A solution may be found in: M. Prażmowska, A proof of the projective Desargues axiom in the desarguesian affine plan, *Demostratio Mathematica*, XXXVII(4), pp. 921–924, 2004.

words 'points' and 'lines' in the proof (and revert incidence accordingly). The resulting text is a proof of φ^* using only axioms $\mathbf{PP_1}^* = \mathbf{PP_2}$, $\mathbf{PP_2}^* = \mathbf{PP_1}$, and $\mathbf{PP_3}^*$. Since the latter is a consequence of $\mathbf{PP_1}$, $\mathbf{PP_2}$, and $\mathbf{PP_3}$, it holds in any projective plane. Therefore so does φ^*. □

12.1.4. Remark. The proof is not very satisfactory as it relies on proof theory. § 12.2 gives a better one.

12.2 The dual plane

Let us now dualise planes.

12.2.1. Definition (dual incidence geometry)**.** Let $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ be an incidence geometry. Set $\mathcal{P}^* = \mathcal{L}$, $\mathcal{L}^* = \mathcal{P}$, and take I^* to be $\ni$. Let $\Gamma^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ be the *dual incidence geometry* of Γ.

12.2.2. Remark (dual incidence as membership)**.** In order to have I^* be $\in$, proceed as follows. For $p \in \mathcal{P}$ let $\mathcal{L}_p = \{\ell \in \mathcal{L} : p \in \ell\}$. Now take $\mathcal{P}^* = \mathcal{L}$, $\mathcal{L}^* = \{\mathcal{L}_p : p \in \mathcal{P}\}$, and $I^* =\in$. We claim that this construction is isomorphic to the previous one.

Introduce better notation: let Γ_1^* be the original construction with I^* and Γ_2^* be the one with $\in$. Let $f : \mathcal{P}_1^* = \mathcal{L} \to \mathcal{L} = \mathcal{P}_2^*$ be the identity map. Now let $g : \mathcal{L}_1^* = \mathcal{P} \to \{\mathcal{L}_p : p \in \mathcal{P}\} = \mathcal{L}_2^*$ take p to $\mathcal{L}_p$. Then letting $p_1^* = \ell \in \mathcal{P}_1^*$ and $m_1^* = q \in \mathcal{L}_1^*$, one has:

$$p_1^* I_1^* m_1^* \quad \Leftrightarrow \quad \ell \ni q \quad \Leftrightarrow \quad \ell \in \mathcal{L}_q \quad \Leftrightarrow \quad f(p_1^*) \in g(m_1^*).$$

12.2.3. Proposition. *Let Γ be an incidence geometry and φ be a sentence. Then $\Gamma \vDash \varphi$ iff $\Gamma^* \vDash \varphi^*$. In particular, Γ is a projective plane iff Γ^* is one.*

Proof. Short, not rigorous: by induction on what a statement is.

Long, rigorous: define elementary formulas *with parameters* $\varphi(\mathbf{a})$. Now a point $p \in \mathcal{P}$ maps to the line $p^* \in \mathcal{L}^*$ and a line $\ell \in \mathcal{L}$ to the point $\ell^* \in \mathcal{P}^*$; so parameters $\mathbf{a}^*$ are well-defined. Then prove by induction on formula with parameters: $\Gamma \vDash \varphi(\mathbf{a})$ iff $\Gamma^* \vDash \varphi^*(\mathbf{a}^*)$. (This is an easy induction.) In particular, a sentence is a formula with no parameters, giving the result. □

Since axioms of a projective plane are self-dual, but axioms of an affine plane are not, we focus on projective planes. (The dual geometry of an affine plane is not an affine plane.)

12.2.4. Remark. In general, $\mathbb{P} \not\simeq \mathbb{P}^*$. However $\mathbb{P} \simeq \mathbb{P}^{**}$ always and canonically.

Better proof of the duality Theorem 12.1.3. Suppose all projective planes satisfy φ. In particular, for all projective planes $\mathbb{P}$, one has $\mathbb{P}^* \vDash \varphi$; hence $\mathbb{P} \vDash \varphi^*$. So all projective planes satisfy φ^*. Applying to φ^*, we get the converse. □

Let $\mathbb{F}$ be a skew-field. Remember that the *opposite* skew-field $\mathbb{F}^{\mathrm{op}}$ has same underlying set, same addition, but multiplication:

$$a \cdot_{\mathrm{op}} b = b \cdot a.$$

It is a skew-field, in general not isomorphic to $\mathbb{F}$.

12.2.5. Remarks.

- There is a general notion of the opposite group G^{op}, and one always has $G^{\mathrm{op}} \simeq G$ (using inversion). But knowing $\mathbb{F}^\times \simeq (\mathbb{F}^\times)^{\mathrm{op}}$ is not enough as addition is not necessarily preserved.
- $\mathbb{F} = \mathbb{F}^{\mathrm{op}}$ iff $\mathbb{F}$ is commutative.
- $\mathbb{H} \simeq \mathbb{H}^{\mathrm{op}}$ thanks to quaternion conjugation. In general, $\mathbb{F} \simeq \mathbb{F}^{\mathrm{op}}$ iff $\mathbb{F}$ admits an 'anti-automorphism'.
- So in general, $\mathbb{F} \not\simeq \mathbb{F}^{\mathrm{op}}$.

12.2.6. Proposition. *Let $\mathbb{F}$ be a skew-field. Then $(\mathbb{P}^2(\mathbb{F}))^* \simeq \mathbb{P}^2(\mathbb{F}^{op})$.*

12.2.7. Remarks.

- In particular, if $\mathbb{F}$ is a commutative field, then $(\mathbb{P}^2(\mathbb{F}))^* \simeq \mathbb{P}^2(\mathbb{F})$.
- The converse may fail. For instance, although $\mathbb{H}$ is not commutative, i.e. $\mathbb{H} \neq \mathbb{H}^{\mathrm{op}}$, one has $\mathbb{H} \simeq \mathbb{H}^{\mathrm{op}}$. So $(\mathbb{P}^2(\mathbb{H}))^* \simeq \mathbb{P}^2(\mathbb{H}^{\mathrm{op}}) \simeq \mathbb{P}^2(\mathbb{H})$.

Proof of the Proposition. One must pay attention to sides. There are three important ideas.

1. The dual of a *left*-vector space is a *right*-vector space.

   This is achieved by letting $(\varphi \cdot \lambda)(v) = \varphi(v) \cdot \lambda$. (Also notice that $(\lambda \cdot \varphi)(v) = \lambda \cdot \varphi(v)$ is *not* a linear form.)

   More generally the usual duality pairing $W \rightsquigarrow W^\perp$ maps left-$\mathbb{F}$-vector subspaces of V to *right*-$\mathbb{F}$-vector subspaces of V^*. (If things are too abstract, one is allowed to think in terms of equations, but then remember that in our convention matrices act from the right. Say elements of V are rows, elements of $V^\perp$ are columns, and one computes $\langle v, \varphi \rangle$ by row-column multiplication, which returns a scalar.)

   With this in mind, duality theory works like in the commutative case.

2. A *right*-vector space over $\mathbb{F}$ is a *left*-vector space over $\mathbb{F}^{\mathrm{op}}$.

   This is achieved by letting $\lambda * v = v \cdot \lambda$. Indeed:

$$\begin{aligned}(\lambda \cdot_{\mathrm{op}} \mu) * v &= (\mu \cdot \lambda) * v \\ &= v \cdot (\mu \cdot \lambda) \\ &= (v \cdot \mu) \cdot \lambda \\ &= \lambda * (\mu * v).\end{aligned}$$

3. $\mathbb{P}^2(\mathbb{F})$ is obtained from $\mathbb{F}^3$ as a *left*-vector space. But $\mathbb{F}^3$ also bears a *right*-vector space structure by letting $(x, y, z) \cdot \lambda = (x\lambda, y\lambda, z\lambda)$. These two actions differ, but commute. Hence $\mathbb{F}^3$ is both a left-vector space and a right-vector space over $\mathbb{F}$, alternatively: both a $\mathbb{F}$ and $\mathbb{F}^{\mathrm{op}}$ left-vector space. But these structures differ as soon as $\mathbb{F}$ is non-commutative.

Let $\mathbb{K} = \mathbb{F}^{\mathrm{op}}$ for clarity. It will save notation to write $\mathbb{P}_\mathbb{F} = \mathbb{P}^2(\mathbb{F})$ and $\mathbb{P}_\mathbb{K} = \mathbb{P}^2(\mathbb{K})$.

If $L \leq \mathbb{K}^3$ is a (left-$\mathbb{K}$-)vector line, then $L^\perp \leq \mathbb{K}^3$ is a (right-$\mathbb{K}$-)vector plane, so $L^\perp \leq \mathbb{F}^3$ is a (left-$\mathbb{F}$-)vector plane. Likewise, if $H \leq \mathbb{K}^3$ is a (left-$\mathbb{K}$-)vector plane, then $H^\perp \leq \mathbb{F}^3$ is a (left-$\mathbb{F}$-)vector line. Now define:

$$f: \begin{array}{ccc} \mathcal{P}(\mathbb{P}_\mathbb{K}) & \to & \mathcal{L}(\mathbb{P}_\mathbb{F}), \\ L & \mapsto & L^\perp \end{array}$$

and:

$$g: \begin{array}{ccc} \mathcal{L}(\mathbb{P}_\mathbb{K}) & \to & \mathcal{P}(\mathbb{P}_\mathbb{F}). \\ H & \mapsto & H^\perp \end{array}$$

These are clearly bijections. Now let $(p, \ell) \in \mathcal{P}(\mathbb{P}_\mathbb{K}) \times \mathcal{L}(\mathbb{P}_\mathbb{K})$, say $p = L \leq \mathbb{K}^3$ and $\ell = H \leq \mathbb{K}^3$. Since duality reverses inclusion:

$$p \in \ell \quad \Leftrightarrow \quad L \leq H \quad \Leftrightarrow \quad H^\perp \geq L^\perp \quad \Leftrightarrow \quad g(\ell) \ni f(p).$$

Therefore $\mathbb{P}_{\mathbb{F}^{\mathrm{op}}} = \mathbb{P}_\mathbb{K} \simeq (\mathcal{L}(\mathbb{P}_\mathbb{F}), \mathcal{P}(\mathbb{P}_\mathbb{F}), \ni) \simeq \mathbb{P}^*_\mathbb{F}$, as desired. □

12.3 Dualising Desargues and Pappus

The Desargues and Pappus properties may be called 'self-dual', but that would be in two distinct senses:

- the Desargues property is equivalent to its dual, meaning that a plane has one iff it has the other;
- the (much stronger) Pappus property implies the (much stronger too) self-duality of a plane, meaning that the pappian plane is isomorphic to its dual.

12.3.1. Proposition (Desargues implies Desargues*)**.** *Let $\mathbb{P}$ be a projective plane. Then $\mathbb{P}$ is desarguesian iff $\mathbb{P}^*$ is.*

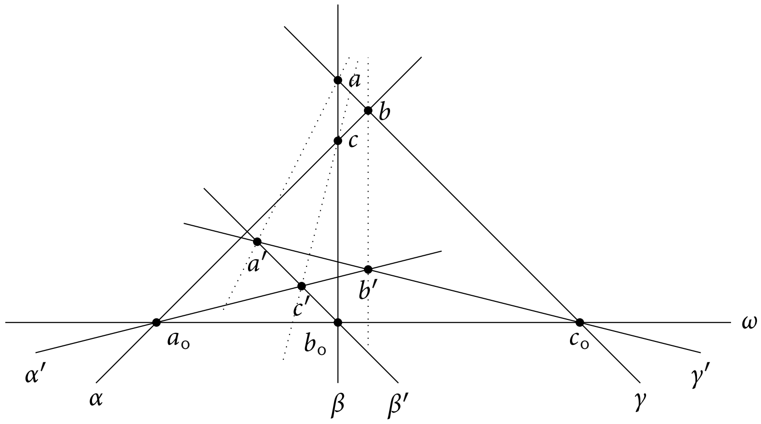
First proof. One implication suffices since $\mathbb{P}^{**} \simeq \mathbb{P}$. Suppose $\mathbb{P}$ is desarguesian. Then using Hilbert coordinatisation, there is a skew-field $\mathbb{F}$ with $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$. Then by Proposition 12.2.6, $\mathbb{P}^* \simeq (\mathbb{P}^2(\mathbb{F}))^* \simeq \mathbb{P}^2(\mathbb{F}^{\mathrm{op}})$ is desarguesian as well. □

Second proof. To prove that $\mathbb{P}^*$ satisfies Desargues, we check that $\mathbb{P}$ satisfies Desargues*. One should first write Desargues*; it so happens that it is the 'converse Desargues property' (Remark 3.1.5).

So let $\omega, \alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be lines such that:

- ω, α, α' are concurrent;
- ω, β, β' are concurrent;
- ω, γ, γ' are concurrent.

To prove Desargues* is to prove that lines $\alpha'' = (\beta \cap \gamma\, \beta' \cap \gamma')$, $\beta'' = (\alpha \cap \gamma\, \alpha' \cap \gamma')$, and $\gamma'' = (\alpha \cap \beta\, \alpha' \cap \beta')$ are concurrent.



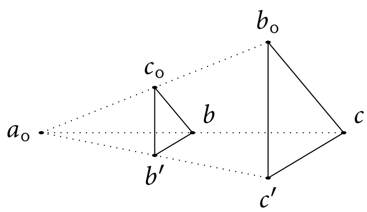
Will dotted lines concur?

We name these points and a few more. Let:

- $a_0 = \alpha \cap \alpha'$, $b_0 = \beta \cap \beta'$, and $c_0 = \gamma \cap \gamma'$;
- $a = \beta \cap \gamma$ and $a' = \beta' \cap \gamma'$;
- $b = \alpha \cap \gamma$ and $b' = \alpha' \cap \gamma'$;
- $c = \alpha \cap \beta$ and $c' = \alpha' \cap \beta'$.

With this notation we now have $\alpha'' = (aa')$, and so on.

Consider the following Desargues configuration (mind the swap on line ω):



The various collinearities occur on lines ω, α, α'. Now compute the 'Desargues points':

- $(c_0 b) = \gamma$ and $(b_0 c) = \beta$ meet at a;
- $(c_0 b') = \gamma'$ and $(b_0 c') = \beta'$ meet at a';
- $(b' b) = \beta''$ and $(c' c) = \gamma''$ meet at say x.

Since $\mathbb{P}$ is desarguesian, $x \in (aa') = \alpha''$. So $\alpha'', \beta'', \gamma''$ concur at x, as desired. □

Assuming the Pappus property, the conclusion can be made even stronger.

12.3.2. Theorem (Pappus implies self-duality). *Let $\mathbb{P}$ be a projective plane. If $\mathbb{P}$ is pappian, then $\mathbb{P}^* \simeq \mathbb{P}$. (In particular $\mathbb{P}^*$ is pappian.)*

Proof. Suppose $\mathbb{P}$ is pappian; we use coordinatisation. By the Pappus property, $\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F})$ for some commutative field $\mathbb{F}$ (Corollary 6.3.2). In particular $\mathbb{F}^{op} = \mathbb{F}$. Therefore taking duals with Proposition 12.2.6:

$$\mathbb{P}^* \simeq \mathbb{P}^2(\mathbb{F})^* \simeq \mathbb{P}^2(\mathbb{F}^{\mathrm{op}}) = \mathbb{P}^2(\mathbb{F}) \simeq \mathbb{P},$$

which is pappian by assumption. □

12.3.3. Remarks.

- There is a geometric proof (without coordinatisation) that if $\mathbb{P}$ is pappian then so is $\mathbb{P}^*$. This is much weaker than Theorem 12.3.2.
- I am not aware of a *geometric* proof that if $\mathbb{P}$ is pappian, then $\mathbb{P} \simeq \mathbb{P}^*$ (one would have to go to self-dualities).
- Be careful that ($\mathbb{P} \simeq \mathbb{P}^*$ and desarguesian) does *not* imply commutativity of the underlying skew-field: one only has $\mathbb{F}^{\mathrm{op}} \simeq \mathbb{F}$, as in the quaternions.

12.4 Exercises

12.4.1. Exercise. *Prove that in an incidence structure satisfying* $\mathbf{PP}_1$ *and* $\mathbf{PP}_2$*, the following are equivalent:*

- *every line has at least three points;*
- *every point is on at least three lines.*

12.4.2. Exercise. *Let* $\mathbb{F}$ *be a skew-field.*

1. *Prove that* $(\mathbb{F})^2\mathbb{P} \simeq \mathbb{P}^2(\mathbb{F}^{op})$ *(see Remark 2.1.4).*
2. *Prove that* $\mathbb{F} \simeq \mathbb{F}^{op}$ *iff* $\mathbb{P}^2(\mathbb{F}) \simeq (\mathbb{P}^2(\mathbb{F}))^*$ *iff* $\mathbb{P}^2(\mathbb{F}) \simeq (\mathbb{F})^2\mathbb{P}$.

12.4.3. Exercise. *Give a geometric proof (in the spirit of Proposition 12.3.1) that Desargues$*$ implies Desargues.*

12.4.4. Exercise. *Give a geometric proof (in the spirit of Proposition 12.3.1) that a projective plane* $\mathbb{P}$ *is pappian iff* $\mathbb{P}^*$ *is.*

Further reading

- E. Artin, *Geometric algebra*. Interscience Publishers, New York/London. 1957.

  The coordinatisation theorem is proved in chapter 2. The whole book is a classic.
- P. Cameron, *Projective and polar spaces*. Queen Mary and Westfield Maths Notes, 13, London. 1992.

  Available on the author's webpage.[21]
- R. Hartshorne, *Foundations of Projective Geometry*. Harvard University Lecture Notes, W. A. Benjamin, Inc., New York. 1967.

  Technically out of print but on the internet one finds a modern typeset.

[21] https://webspace.maths.qmul.ac.uk/p.j.cameron/pps/

- D. Hilbert, *Grundlagen der Geometrie*. B. G. Teubner, Leipzig. 1899.

  One may find a translation of this historical text online.[22] Chapter 5 (on Desargues' Theorem) is the origin of the topic.

- D. Hughes and F. Piper, *Projective planes*. Graduate texts in mathematics, 6. Springer-Verlag, New-York/Berlin. 1973.

  Recommended reference on projective planes.

- H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel, *Compact projective planes, with an introduction to octonion geometry*. de Gruyter Expositions in Mathematics, 21. 1995.

  Adding back topological ingredients, a complete study of the projective plane over the octonions.

[22] https://math.berkeley.edu/~wodzicki/160/Hilbert.pdf