Very introductory notes of Category Theory

Free distributed copy - Mistakes included[1]

Version 2.01 beta - July 19, 2021

A possible motivation

We usually all start learning that there are things called numbers, like 0, 1, 3, 5, 7, 31, -53, 2027 etc. They teach us also that we can add them, multiply them, and we spend time learning addition and multiplication tables. Then we learn that this is an example of what is called a semigroup, a group, a ring or a field, according to what and how we look at them. As examples of groups that are 'new', we then are given often integers modulo some n, polynomials or matrices. They serve as motivation for the extra job of abstraction, as well as to show other contexts where things happen "as if they were normal numbers, except they are not". Somehow, we will now do the same: we will abstract from the structures you have seen so far and those things will be used as particular examples.

Why these notes

The reason of these notes is not to be a reference for or a detailed discussion on Category Theory. For this, there are already excellent books available. The intention of this exposition is to give an idea of the most fundamental concepts of this area and examples on how to work with categories. The style in particular tries to be as much as possible as it would be in a lecture. Because of all these aspects, proofs are either given to showcase the techniques, or left as exercise, when doable and useful, or omitted with a reference to other more serious textbooks.

1 Starting definitions

An apology: as anyone (!) should know, it is very easy to construct paradoxes and put them in a mathematical form. This, for example, is Russell's paradox, a formal version of the barber paradox:

$$A = \{B \mid B \notin B\}, \qquad A \in A \ldots ?$$

To avoid this, very roughly speaking, we just set rules that forbid ourselves to talk about the sets of se..., because it is not a set. We will want to talk about the category of sets though, hence some framework to deal with this, avoiding (hopefully... thanks Gödel) contradictions, is needed. This can and has been done in several ways. So we won't. The IR[2] can check Kashiwara and Shapira's book *Categories and Sheaves* [1] or the famous Mac Lane's book *Categories for the working mathematician* [2] for a short and safe framework. And now finally:

Definition 1.1 *A category $\mathcal{C}$ is the following set of data:*

1. *A set, denoted by* $\mathrm{Obj}(\mathcal{C})$*;*
2. *For every ordered pair X, Y in $\mathrm{Obj}(\mathcal{C})$, a set, denoted by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$;*
3. *For every ordered tern X, Y, Z in $\mathrm{Obj}(\mathcal{C})$, a function*

$$\circ^{\mathcal{C}}_{XYZ} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

with the following properties:

(a) *For each element X in $\mathrm{Obj}(\mathcal{C})$ there exists an element $\mathbb{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, such that for any f in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ we have*

$$\circ^{\mathcal{C}}_{XXY}(\mathbb{1}_X, f) = \circ^{\mathcal{C}}_{XYY}(f, \mathbb{1}_Y) = f;$$

(b) *For every $f \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $h \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we have*

$$\circ^{\mathcal{C}}_{WYZ}\left(\circ^{\mathcal{C}}_{WXY}(f,g), h\right) = \circ^{\mathcal{C}}_{WXZ}\left(f, \circ^{\mathcal{C}}_{XYZ}(g,h)\right) \ .$$

[1] 1% of mistakes is intentional.

[2] Interested Reader.

Various remarks 1.2

- *It should be clear that the definition is not that clear, for now.*
- *To ease the notation and make it more intuitive, but* only *for this reason, we will*
  - *Call the elements of* $\mathrm{Obj}(\mathcal{C})$ objects *and write* $X \in \mathcal{C}$ *instead of* $X \in \mathrm{Obj}(\mathcal{C})$ *when possible;*
  - *Call the elements of* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ morphisms *or* arrows*, but NEVER functions (unless when they are). We will also write* $f : X \to Y$ *or* $X \xrightarrow{f} Y$ *to mean* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$*;*
  - *As a technical remark, we should also state that* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \cap \mathrm{Hom}_{\mathcal{C}}(V,W) = \emptyset$*. That means that a morphism is such only for a single (ordered) pair of objects;*
  - *We will never*[3] *again write* $\circ^{\mathcal{C}}_{XYZ}(f,g)$*, we will write* $g \circ^{\mathcal{C}}_{XYZ} f$ *instead. Usually, we will also drop the indexes and will read the formula* $g \circ f$ *as "composition of* g *and* f*" or "*g *after* f*";*
  - *The morphism* $\mathbb{1}_X$ *will be called* the identity of X*;*
  - *We will sometimes refer to the axiom (1.1.(3b)) as* associativity of composition*.*
- *Please take a moment to observe how much space, in the definition, is reserved to the objects and how much is reserved to the morphisms. This can be pushed even further, see [2, p.9] for a definition of a (meta)category without mentioning objects. This last statement might be surprising; to make it sound less magical, please realize that in practice the objects serve only as labelling on morphisms to tell us when we can compose two of them. In any case, from now on, your attention to the two concepts should be shared proportionally.*

With the above remarks, the axioms of a category become then, for example: for every $X \in \mathcal{C}$ there exists a morphism $\mathbb{1}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called *identity on* X, such that, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$

$$f \circ \mathbb{1}_X = \mathbb{1}_Y \circ f = f.$$

Exercise 1.3 *Rewrite property (3) in Definition (1.1) with the simplified notation above.*

These last formulae should sound more reassuring, but let us explicitly remark that:

- The objects of $\mathcal{C}$ are NOT sets;
- The elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are NOT functions;
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- The objects of $\mathcal{C}$ are NOT sets;
- The elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are NOT functions.

[3] almost

And now examples (various explanations and remarks can be found afterwards):

Category	Objects	Morphisms
$\mathcal{S}$et	sets	$\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X,Y) = \{$functions from X to $Y\}$
$\mathcal{G}$r	groups	$\mathrm{Hom}_{\mathcal{G}\mathrm{r}}(X,Y) = \{$group homomorphisms from X to $Y\}$
$\mathcal{A}b$	abelian groups	$\mathrm{Hom}_{\mathcal{A}b}(X,Y) = \{$group homomorphisms from X to $Y\}$
R-mod, with R a ring	(left) modules on R	$\mathrm{Hom}_R(X,Y) =$ $\{$homomorphisms of (left) R modules from X to $Y\}$
Vect_k	k-vector spaces	$\mathrm{Hom}_{\mathrm{Vect}_k}(V,W) = \mathrm{Hom}_k(V,W) =$ $\{k$-linear maps from V to $W\}$
$\mathcal{T}$op	topological spaces (X,τ_X)	$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X,Y) = C(X,Y) =$ $\{$continuous functions from X to $Y\}$
$\mathcal{T}\mathrm{op}_\bullet$	topological spaces with a distinguished point (X,τ_X,x)	$\mathrm{Hom}_{\mathcal{T}\mathrm{op}_\bullet}((X,\tau_X,x),(Y,\tau_Y,y)) =$ $\{$continuous functions $f : X \to Y$ s.t. $f(x) = y\}$
$(S,\leq)$ a poset	the elements of S	Given $s,t \in S$, $\mathrm{Hom}_S(s,t) = \begin{cases} \{*\} & \text{if } s \leq t, \\ \emptyset & \text{otherwise} \end{cases}$
$(M,\cdot)$ a monoid	the singleton $\{*\}$	$\mathrm{Hom}_M(*,*) = M$, with composition given by the product $\cdot$ in M
$\mathcal{P}$oset	partially ordered sets	order preserving functions

The philosophy behind the choice of what the morphisms should be, could be explained as follows: the arrows from an object to another are the type of transformations we decide to allow / we are interested in.

We invite the reader (IR or not) to check that for each category listed the axioms are indeed satisfied. We will explicitly work out the more interesting or less intuitive examples in the table above.

$\mathcal{S}$et Yes, in this case the objects are indeed sets. It may not be necessary to explain why the arrows we naturally consider are actually functions. It might be worth though to add a remark: in theory, the sets $\{1\}$, $\{549875\}$, $\{a\}$, $\{$category$\}$, and so on, are different, but basically they are the same, having only one element (see Definition (1.6) and Example (1.7.1)). Each of these sets is called a *singleton* and we will generally denote any of them by $\{*\}$.

$\mathcal{A}b$ This one needs no explanation. Let's remark explicitly only that this is the other theory, other than $\mathcal{S}$et, upon which Category Theory is modeled. These are the examples one should never forget.

$\mathcal{T}$op In the case of $\mathcal{T}$op, the objects are, let's say, sets X with an extra structure τ_X, that is the topology. Hence, we are interested in functions that behave 'well', so to say, towards the topology. It turns out that, given topological spaces (X,τ_X) and (Y,τ_Y), the functions f, such that $f^{-1}(V)$ is in τ_X for any element V in τ_Y, are very interesting. These are called *continuous*, and here is $\mathcal{T}$op.

$(S,\leq)$ This is a very good example to see that morphisms are, indeed, not necessarily functions. To check that this realizes a category, with the construction given in the table above, we should first of all define what $\circ^S$ should be. In order to do that, we need two morphisms to compose, but this already means that we need $s,t,u \in S$ such that $\mathrm{Hom}_S(s,t)$ and $\mathrm{Hom}_S(t,u)$ are non-empty, otherwise there would be nothing to compose. For those morphism sets, to be non-empty means actually, by definition, that $s \leq t$ and $t \leq u$; moreover, by construction, for each set $\mathrm{Hom}_S(s,t)$ and $\mathrm{Hom}_S(t,u)$ there is only one morphism. Given this (unique) pair, the question is: do we have any choice for the composition? That is, again by definition: is $\mathrm{Hom}_S(s,u)$ non-empty? Yes, because, by transitivity, $s \leq t$ and $t \leq u$ does imply $s \leq u$, so we do have a morphism to choose, and that is the only one possible as well. So a composition is defined and it is unique. By the same reason, we also have identity morphisms, that is the only morphisms in the sets $\mathrm{Hom}_S(s,s)$, that are non-empty by reflexivity.

$(M, \cdot)$ If in the previous example we had no restrictions on the quantity of objects but, somehow, we had the minimum possible amount of morphims (the actual minimum would be having only the identities, that would give a so called *discrete category*), the case given by a monoid is the other extreme: a category with only one object and no restriction on the morphisms. Given that we have only one object, the axioms need to be checked for only one morphism set, that is $\mathrm{Hom}_M(*, *) = M$. Finally, the properties we ask from the composition reduce exactly to the properties of multiplication in a monoid: it is associative and there exists an (algebraic) identity 1_M that behaves as wanted.

It must be highlighted once more that, so far, we have not proved anything new. In all the previous discussions there is no, let's say, new knowledge created: we have *just* reinterpreted facts we already knew in a different language.

As an easy practice, you can solve the following

Exercise 1.4 *Prove that, given a category* $\mathcal{C}$*, in each* $\mathrm{Hom}_{\mathcal{C}}(X, X)$ *there is a unique identity.*

Let's conclude this section with a general recommendation:

To test your understanding of any following new concept,
try it on (the category associated to) a poset or a monoid.

1.1 Diagrams

To help our intuition in the following discussions, we introduce a graphical notation for morphisms and compositions: we will call a *diagram* an oriented graph where the vertices are objects of a category $\mathcal{C}$ and the edges are morphisms between the objects represented, with the obvious orientation. Long story short, a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ will be graphically represented by

$$X \xrightarrow{f} Y \; .$$

Sometimes we could also omit the labels and have

$$\bullet \xrightarrow{f} \bullet$$

or, for example,

$$\bullet \underset{f}{\overset{g}{\rightrightarrows}} \bullet \; ,$$

the latter being a way to denote two morphisms f and g with the same source and destination (and hence referred as to be *parallel*).

Walking through a path of such a graph, is done via composition of morphisms, or in other words, given the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \; ,$$

we say that *we can go from X to Z via $g \circ f$.*

We then say that a diagram *commutes* if any two paths between two vertices are equal. So for instance, by definition of composition, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g \circ f} & \downarrow g \\ & & Z \end{array} \; .$$

We can also rephrase as commutative diagrams the properties of composition in a category. For example, associativity translates into the commutativity of this diagram:

$$\begin{array}{ccc} W & \xrightarrow{g \circ f} & Y \\ f \downarrow & & \downarrow h \\ X & \xrightarrow[h \circ g]{} & Z \end{array} \; .$$

Exercise 1.5 *Sketch (part of) the graph associated to the category associated to the poset* $(\mathbb{N}, \leq)$*, where* $a \leq b$ *if and only if* a *divides* b*.*

We are ready to give our first definition inside a category:

Definition 1.6

- *A morphism* $f : X \to Y$ *in a category* $\mathcal{C}$ *is an* isomorphism *if there exists a morphism* $g : Y \to X$ *such that* $g \circ f = \mathbb{1}_X$ *and* $f \circ g = \mathbb{1}_Y$*. Such a morphism* g *is normally denoted by* f^{-1} *and it is called an* inverse *of* f*.*

- *If there exists an isomorphism* $f : X \to Y$*, then* X *and* Y *are said to be* isomorphic*.*

It is immediate to prove that an inverse, if it exists, is unique: assume g_1 and g_2 are inverses of $X \xrightarrow{f} Y$, then

$$g_1 = g_1 \circ \mathbb{1}_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathbb{1}_X \circ g_2 = g_2.$$

Hence, we can from now on talk about *the* inverse of f, whenever it exists, and the notation f^{-1} is justified.

Example 1.7

1. *In the category* $\mathcal{S}$et*, a morphism, that is a function, is an isomorphism if and only if it is has an inverse, that is if and only if the function is invertible, that is if and only if it is a bijection. No surprises here.*

2. *In* $\mathcal{T}$op*, an isomorphism is a bijection that, being a morphism in the category, is continuous and open (or its inverse is continuous). These are the well known homeomorphisms.*

3. *If* $(M, \cdot)$ *is a monoid considered as a category, then a morphism* $m \in M = \mathrm{Hom}_M(*, *)$ *is an isomorphism if and only if it is invertible as an element of the monoid.*

One should already remark that the definition of an isomorphism is just what it is. It gives a guideline to define what they are. Each field of study will have then a list of tools to describe isomorphisms in their own language. For example, the simple property of being bijective makes no sense in a category, since in general objects do not have elements. This is indeed part of the game: redefine concepts in terms not of elements but morphisms.

Before continuing on the path of abstraction, let us see how this game can be played with two more examples. Before though, let us introduce an apparently innocuous but very useful definition, or better, a notation:

Definition 1.8 *Given a category* $\mathcal{C}$*, we denote by* $\mathcal{C}^{\mathrm{op}}$ *the* opposite category of $\mathcal{C}$ *described by*

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$*;*

- *For any pair of objects of* $\mathcal{C}^{\mathrm{op}}$*, we have* $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$*;*

- $\circ^{\mathcal{C}^{\mathrm{op}}}_{XYZ} = \circ^{\mathcal{C}}_{ZYX}$*.*

Remember that, once again, morphisms in a category are not functions but just elements of a set associated to the pair of objects (X, Y) in the cartesian product $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C})$. To define $\mathcal{C}^{\mathrm{op}}$ then, we just find the sets associated to the pairs (X, Y) and (Y, X) and switch them. Verifying that $\mathcal{C}^{\mathrm{op}}$ is indeed a category, is a (really) trivial exercise:

Exercise 1.9 *Given a category* $\mathcal{C}$*, prove that*

1. $\mathcal{C}^{\mathrm{op}}$ *is a category, with* $\mathbb{1}_X$ *in* $\mathcal{C}$ *equal to* $\mathbb{1}_X$ *in* $\mathcal{C}^{\mathrm{op}}$*;*

2. $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$*;*

3. *There is an obvious functor* $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ *(see Definition (2.1)).*

We will see more of how to play inside a category in Section 3, but as an early example on how to use $\mathcal{C}^{\mathrm{op}}$, we will work on the the following

Definition 1.10 *A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ in a category $\mathcal{C}$ is*

- *a* monomorphism *if, for any $g,h \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, we have*

$$f \circ g = f \circ h \qquad \Longleftrightarrow \qquad g = h,$$

  that is, whenever f can be canceled on the left;

- *an* epimorphism *if, for any $m,n \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we have*

$$m \circ f = n \circ f \qquad \Longleftrightarrow \qquad m = n,$$

  that is, whenever f can be canceled on the right.

Example 1.11

- *In $\mathcal{S}$et, f is a monomorphism (as a morphism) if and only if it is injective (as a function). Let us prove this explicitly: let f be an element of $\mathrm{Hom}_{\mathcal{S}\mathrm{et}}(X,Y)$, now*

  ($\Rightarrow$) for any $x_1, x_2 \in X$, define the maps $g_i : \{*\} \to X$ so that $g_i(*) = x_i$ for $i = 1,2$. Then, if $f(x_1) = f(x_2)$, we have the following chain of implications:

$$f \circ g_1(*) = f(x_1) = f(x_2) = f \circ g_2(*) \quad \Rightarrow \quad f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2 \quad \Rightarrow \quad x_1 = g_1(*) = g_2(*) = x_2,$$

  and hence, by definition, f is injective.

  ($\Leftarrow$) Let $g_1, g_2 \in \mathrm{Hom}(W,X)$ such that $f \circ g_1 = f \circ g_2$. For any $w \in W$, we then have $f \circ g_1(w) = f \circ g_2(w)$ and since f is assumed to be injective, this means $g_1(w) = g_2(w)$, that is $g_1 = g_2$.

- ***Exercise 1.12*** *Take a ring $(R,+,\cdot)$ and consider $(R,+)$ as a left $R-$module.*

  1. *Show that, for every element $r \in R$, the multiplication by r on the left, denoted by $[r]$, is a morphism in $R-$mod;*
  2. *Show that the morphism $[r]$ is a monomorphism if and only if r is not a zero divisor;*
  3. *When is $[r]$ an isomorphism?*
  4. *Can $[r]$ be an epimorphism without being an isomorphism?*

Exercise 1.13 *Prove that a function f is an epimorphism in $\mathcal{S}$et if and only if it is surjective.*

Exercise 1.14 *What are the epimorphisms in the category of Hausdorff topological spaces?*

From the experience of the 'regular' Mathematics, the following proposition should not be surprising:

Proposition 1.15 *In a category $\mathcal{C}$, isomorphisms are both monomorphisms and epimorphisms.*

Proof: Let $f : X \to Y$ be an isomorphism and $g,h \in \mathrm{Hom}_{\mathcal{C}}(W,X)$ and $m,n \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ morphisms such that $f \circ g = f \circ h$ and $m \circ f = n \circ f$. Then we have

$$m = m \circ \mathbb{1}_Y = m \circ (f \circ f^{-1}) = (m \circ f) \circ f^{-1} = (n \circ f) \circ f^{-1} = n \circ (f \circ f^{-1}) = n \circ \mathbb{1}_Y = n;$$

and the same for g and h.

q.e.d.

Exercise 1.16 *If a morphism f in a category $\mathcal{C}$ is both a monomorphism and an epimorphism, is it also an isomorphism?* (Hint: no.)

We can now use Definition (1.8) to relate the two concepts:

Proposition 1.17 *A morphism f in a category $\mathcal{C}$ is*

1. *an isomorphism if and only if it is an isomorphism in $\mathcal{C}^{\mathrm{op}}$;*
2. *a monomorphism if and only if it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.*

Proof: For the first statement, if $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, then f is an isomorphism if and only there exists a morphism $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that

$$f \circ^{\mathcal{C}}_{XYX} f^{-1} = \mathbb{1}_X \quad \text{and} \quad f^{-1} \circ^{\mathcal{C}}_{YXY} f = \mathbb{1}_Y.$$

Re-writing these equalities in C^{op}, and remembering Exercise (1.9), the condition becomes: if and only if there exists a morphism $f^{-1} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y)$ such that

$$f^{-1} \circ^{\mathcal{C}^{\mathrm{op}}}_{XYX} f = \mathbb{1}_X \quad \text{and} \quad f \circ^{\mathcal{C}^{\mathrm{op}}}_{YXY} f^{-1} = \mathbb{1}_Y.$$

For the second part, we have

$$\begin{array}{ccccc} f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \text{ mono} & \Leftrightarrow & \begin{array}{c} \forall g,h \in \mathrm{Hom}_{\mathcal{C}}(W,X), \\ f \circ^{\mathcal{C}}_{WXY} g = f \circ^{\mathcal{C}}_{WXY} h \Rightarrow g = h \end{array} & & \\ & & \Updownarrow & & . \\ & & \begin{array}{c} \forall m,n \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,W), \\ m \circ^{\mathcal{C}^{\mathrm{op}}}_{YXW} f = n \circ^{\mathcal{C}^{\mathrm{op}}}_{YXW} f \Rightarrow m = n \end{array} & \Leftrightarrow & f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,X) \text{ epi} \end{array}$$

q.e.d.

Corollary 1.17.1 *Convince yourself that this completes the part of proof of Proposition (1.15) that we did not write explicitly.*

More in general, it is common to say that a property P of an object or a morphism is the *dual* of a property Q if it is verified if and only if Q is verified in $\mathcal{C}^{\mathrm{op}}$ for the same object or morphism.

2 Functors

We have defined some new entities, that is categories, but then again: to be faithful to the principle that what counts are not the objects (in a general sense) but the transformations among them, we should have a corresponding concept for categories as well. So to say, we need to define what would be the morphisms between categories:

Definition 2.1 *Given two categories $\mathcal{C}$ and $\mathcal{D}$, a* functor $F : \mathcal{C} \to \mathcal{D}$ *is the following data:*

1. *A function $\tilde{F} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ (see Notation (2.2));*
2. *For every X and Y in $\mathrm{Obj}(\mathcal{C})$, a function $F_{XY} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(\tilde{F}(X), \tilde{F}(Y))$ such that*
   (a) *For every X in $\mathcal{C}$, we have $F_{XX}(\mathbb{1}_X) = \mathbb{1}_{F(X)}$;*
   (b) *For every $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y,Z)$, we have $F_{XZ}(g \circ^{\mathcal{C}} f) = F_{YZ}(g) \circ^{\mathcal{D}} F_{XY}(f)$. In terms of commutative diagrams:*

$$\begin{array}{ccc} \tilde{F}(X) & \xrightarrow{F_{XY}(f)} & \tilde{F}(Y) \\ & \circlearrowleft \; \searrow^{F_{XZ}(g \circ f)} & \downarrow F_{YZ}(g) \\ & & \tilde{F}(Z) \end{array} .$$

For historical reasons, functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is referred to as a contravariant functor *from $\mathcal{C}$ to $\mathcal{D}$, the reason being that it 'reverses the direction of the arrows'.*

In some sense, a functor is a way to translate $\mathcal{C}$ in $\mathcal{D}$, compatibly with the identities and the composition.

Notation 2.2 *We will soon stop writing $\tilde{F}$ and F_{XY} and start writing just F for all of them.*

Example 2.3

1. *The first family of examples of functors is the one of the* forgetful functors, *that is a generic term to indicate a situation where 'structure' is, indeed, forgotten:*

(a) $\rho : \mathcal{G}\mathrm{r} \to \mathcal{S}\mathrm{et}$ *associates to a group* $(G, \cdot)$ *the underlying set* G*; to a group homomorphism* $f : (G, \cdot) \to (H, \star)$ *it associates the same* f*, but seen just as a function between the sets* G *and* H*.*

(b) $\rho : \mathcal{A}b \to \mathcal{G}\mathrm{r}$ *associates to an abelian group, itself, forgetting that the group is abelian.*

(c) $\rho : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ *associates to a topological space* (X, τ_X) *the underlying set* X*; to a continuous function* $f : (X, \tau_X) \to (Y, \tau_Y)$ *it associates the same* f*, but seen just as a function between the sets* X *and* Y*.*

(d) $\rho : \mathrm{Vect}_k \to \mathcal{A}b$ *associates to a vector space* $(V, +, \cdot)$ *the underlying abelian group* $(V, +)$*; to a linear map* $f : V \to W$ *it associates the same* f*, but seen just as a group homomorphism between the groups* $(V, +)$ *and* $(W, +)$*.*

(e) ***Exercise 2.4*** *Define your own forgetful functor.*

(f) ***Exercise 2.5*** *Show that, forgetting the space* X *instead of the topology, you can have also a forgetful functor* $\rho : \mathcal{T}\mathrm{op} \to \mathcal{P}\mathrm{oset}$*.*

2. *We now define a functor* F *from* $\mathcal{S}\mathrm{et}$ *to* $\mathcal{G}\mathrm{r}$*. Given a set* X*, we define* $F(X)$ *as the* free group over the set X*. It is described by the presentation* $< X \mid \emptyset >$*. More explicitly, the group* $F(X)$ *is the quotient set of words written in the alphabet* $X \cup X^{-1}$ *over an equivalence relation* $\sim$*, where*

   - *the set* X^{-1} *is the set of symbols* x^{-1} *for each* $x \in X$*,*
   - *the composition between words is juxtaposition,*
   - *the equivalence relation is given by "equality up to substitution* $xx^{-1} = x^{-1}x = \emptyset$*"; that is, for example* $ac \sim ab^{-1}bc \sim pp^{-1}ab^{-1}bc$*.*
   - *The symbol* $\emptyset$ *denotes here the empty word and its equivalence class is the identity of* $F(X)$*.*

   Given then a function $f : X \to Y$*, we need to define a group homomorphism* $F(f) : F(X) \to F(Y)$*. We do it by alphabetic substitution: we fix the images* $f(x^{-1}) = f(x)^{-1}$*, and the image of (the class of) a word* $x_1x_2 \ldots x_n$ *is (the class of) the word* $f(x_1)f(x_2) \ldots f(x_n)$*. We hence concluded the definition of* F*. We will return on this example in Example (7.7.1).*

3. $\pi_1 : \mathcal{T}\mathrm{op}_\bullet \to \mathcal{G}\mathrm{r}$ *associates to a topological space* (X, τ_X, x) *with a distinguished point, the fundamental group of* X *at* x*. The IR can check any textbook mentioning Algebraic Topology (a standard reference is [3]). This functor is very important because it allows a classification of topological spaces.*

4. *Let* G *be a group and consider the category associated to it, as for the monoid* M*. An action of* G *on an abelian group* A *is then a functor from* G *to* $\mathcal{A}b$ *such that the image of* $\{*\}$ *is* A*.*

5. *An entire, and unavoidable, class of functors, on which we will return several times (see for example section (5)), is constructed as follows:*

   let $\mathcal{C}$ *be a category and* $X \in \mathrm{Obj}(\mathcal{C})$*, we have* $h_\mathcal{C}(X)$ *and* $k_\mathcal{C}(X)$ *defined as follows:*

$$\begin{array}{rccc} h_\mathcal{C}(X): & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{S}\mathrm{et} \\ & Y & \longmapsto & \mathrm{Hom}_\mathcal{C}(Y, X), \end{array}$$

$$\begin{array}{rccc} k_\mathcal{C}(X): & \mathcal{C} & \longrightarrow & \mathcal{S}\mathrm{et} \\ & Y & \longmapsto & \mathrm{Hom}_\mathcal{C}(X, Y). \end{array}$$

   On the morphisms, they are defined as follows:

$$\begin{array}{cccccccc} & \text{in } \mathcal{C} & & \text{in } \mathcal{C}^{\mathrm{op}} & & \text{in } \mathcal{S}\mathrm{et} & & \\ & W & & W & & \mathrm{Hom}_\mathcal{C}(W, X) & \alpha & \\ h_\mathcal{C}(X): & \uparrow f & = & \downarrow f & \longmapsto & \downarrow f_* & \downarrow & , \\ & W' & & W' & & \mathrm{Hom}_\mathcal{C}(W', X) & \alpha \circ f & \end{array}$$

$$\begin{array}{cccccc} & \text{in } \mathcal{C} & & \text{in } \mathcal{S}\mathrm{et} & & \\ & Y & & \mathrm{Hom}_\mathcal{C}(X, Y) & \alpha & \\ k_\mathcal{C}(X): & \downarrow g & \longmapsto & \downarrow g^* & \downarrow & . \\ & Y' & & \mathrm{Hom}_\mathcal{C}(X, Y') & g \circ \alpha & \end{array}$$

Let's take a moment to understand this example, since it is a key step to the whole Category Theory (no exaggeration).

The starting category is X. The target category is just $\mathcal{S}\mathrm{et}$: indeed we defined in (1.1.2) the morphisms as elements of a fixed set denoted by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (remember that they are not*, once more,* functions*). So at least the definition should make sense. We work now with $k_{\mathcal{C}}$, but the same applies to $h_{\mathcal{C}}$.*

Then, by the definition of a functor given above (2.1), given any two objects of the first category (here Y and Y', in $\mathcal{C}$), and found their corresponding images via F (here by contruction $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y')$), for any morphisms from Y to Y' in $\mathcal{C}$ (here g) we need to define a morphim in the target category between their images $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y')$... and now finally *$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y')$ are* actual *sets and a morphism is an* actual *function, while the elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y')$ are morphism (of objects in $\mathcal{C}$). So, using the morphism $g: Y \to Y'$ (that is not a function), we need to find a function that associates to each morphism $\alpha: X \to Y$ a morphism $X \to Y'$. Looking at g and α, it should be natural trying to compose them. The only way to do that is to consider $g \circ \alpha$. By definition (see again (1.1.3)), this is a morphism from X to Y' and hence, by definition, indeed an element of $\mathrm{Hom}_{\mathcal{C}}(X,Y')$. In conclusion, the construction does define a functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$ (after you have solved the following exercise).*

Exercise 2.6 *We leave it to the reader to verify the conditions in Definition (2.1) for $k_{\mathcal{C}}$.*

Exercise 2.7 *Verify Definition (2.1) for $h_{\mathcal{C}}$.*

Exercise 2.8

1. *Show that a category $\mathcal{S}$, such that $\mathrm{Hom}_{\mathcal{S}}(s,t)$ has only 0 or 1 elements, corresponds to a pre-ordered set $(S, \leq)$ (that is, $\leq$ is a partial order without antisymmetry).*
2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two categories as above. To what corresponds a functor $F: \mathcal{S} \to \mathcal{T}$?*

Remark 2.9 *It should not be surprising that*

1. *Functors can be composed: functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}''$ give a functor $G \circ F: \mathcal{C} \to \mathcal{C}''$,*
2. *The process of, say, 'doing nothing' is a functor $\mathbb{1}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$.*

Exercise 2.10 *Properly define and prove the statements in Remark (2.9).*

Exercise 2.11 *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. Show that if f is an isomorphism in $\mathcal{C}$, then also $F(f)$ is an isomorphism in $\mathcal{D}$.*

After all the job and marketing we have done, the following definition shall not be unexpected:

Definition 2.12 *We denote by $\mathcal{C}\mathrm{at}$ the category whose objects are categories and where the morphisms between two of its objects $\mathcal{C}$ and $\mathcal{D}$ are the functors from $\mathcal{C}$ to $\mathcal{D}$. Instead of writing $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C},\mathcal{D})$, we denote the latter by $\mathrm{Fct}(\mathcal{C},\mathcal{D})$.*

There are now two directions in which we can continue exploring the subject. One is to work with the property of objects or, better, of morphisms inside a given category; the other is study relations between categories. We will follow both, in the given order.

3 Working inside a category

In this section, we will try to do many different things at the same time. We will define fundamental types of morphisms and objects, stressing on the new style of giving definition and, consequently, on the new techniques to prove properties. In general, at least in this introduction, the game will be to redefine well known objects (in a generic sense this time) in the context of a category. In practice, this means: without referring to 'elements in a set'. We have actually already seen the example of monomorphisms and epimorphisms, that generalizes respectively injective and surjective functions. Let us push that approach one step further. Specifically, we all know that, whenever we have a subset T of a set S or a subgroup H of a group G, we have a natural injection $\iota: T \hookrightarrow S$ or $\iota: H \hookrightarrow G$. Again, this relies on having specific elements of S, or G, to belong to the sub-object T or H. We want to get rid of the elements, and this is a way to do it:

Definition 3.1 *Let X be an object in a category $\mathcal{C}$ and $\iota_1 : X_1 \to X$ and $\iota_2 : X_2 \to X$ be two monomorphisms. We say that ι_1* is equivalent to *ι_2 if there exists an isomorphism $k : X_1 \to X_2$ such that $\iota_1 = \iota_2 \circ k$. We denote this by $\iota_1 \sim \iota_2$.*

In diagrams, this is equivalent to say that the there exists an isomorphism h such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\iota_1} & X \\ & \overset{\exists k}{\searrow} & \uparrow \iota_2 \\ & & X_2 \end{array} .$$

It is clear (exercise) that $\sim$ is an equivalence relation. Now we are ready to give the following:

Definition 3.2 *A* subobject *Y* of an object *X* in a category *$\mathcal{C}$ is an equivalence class of monomorphisms to X.*

Let us expand this definition with a simple example:

Example 3.3 *We want to find the subobjects of $\mathbb{Z}/6\mathbb{Z}$ in $\mathcal{A}b$. Hence, we need first to list the possible monomorphisms to $\mathbb{Z}/6\mathbb{Z}$, that is, in this case, injections. Since everything is cyclic, in order to define (homo)morphisms we only need to say where 1 is sent:*

$$\varphi_1 : \begin{array}{ccc} \mathbb{Z}/3\mathbb{Z} & \to & \mathbb{Z}/6\mathbb{Z} \\ 1 & \mapsto & 2 \end{array}, \qquad \varphi_2 : \begin{array}{ccc} \mathbb{Z}/3\mathbb{Z} & \to & \mathbb{Z}/6\mathbb{Z} \\ 1 & \mapsto & 4 \end{array}, \qquad \psi : \begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/6\mathbb{Z} \\ 1 & \mapsto & 3 \end{array}.$$

Finally, it is clear that $\varphi_1 \sim \varphi_2$, because the map $k : 1 \mapsto 2$ is an isomorphism from $\mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ (being its own inverse) such that $\varphi_2 = \varphi_1 \circ k$. So we have two subobjects of $\mathbb{Z}/6\mathbb{Z}$: one is the class with two elements $\{\varphi_1, \varphi_2\}$, that corresponds to the subgroup of order 3 listed as $\{0, 2, 4\}$, and the other is the class $\{\psi\}$, that corresponds to the subgroup of order 2 listed as $\{0, 3\}$.

Using a more abstract notation to help forget about numbers, this expresses the following facts:

1. *We want to pass from talking of a subgroup, to talking of the embedding ι associated to that subgroup.*
2. *Already in the example of the cyclic group C_6 though (say with generator σ), we have that the cyclic group C_3 (say with generator δ) embeds in two different ways, that are the ones given by sending δ to each of the two elements of C_6 that have order 3, that is σ^2 and σ^4.*
3. *Clearly though, C_6 has a unique subgroup of order 3, so two embeddings make no sense to us.*
4. *Luckily, or indeed, they are equivalent, because their image is the same, so we have a unique equivalence class, that represents the unique subgroup.*

Exercise 3.4 *Describe the subobjects of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (or, if you are braver, of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$).*

Exercise 3.5 *In the same way as in Definition (3.2), define the quotients of an object X in a category $\mathcal{C}$. You can check if your solution is correct by verifying that P is a quotient of X in $\mathcal{C}$ if and only if it is a subobject of X in $\mathcal{C}^{\text{op}}$.*

It is now the moment to highlight another aspect of the definition of a monomorphism: it describes the morphisms we want to define in relation to other morphisms. In a similar fashion, we give the following

Definition 3.6 *Let $\mathcal{C}$ be a category.*

- *An object T in $\mathcal{C}$ is called* terminal *if for every object X in $\mathcal{C}$ we have* $\text{Hom}_{\mathcal{C}}(X, T) = \{*\}$.
- *An object I in $\mathcal{C}$ is called* initial *if for every object X in $\mathcal{C}$ we have* $\text{Hom}_{\mathcal{C}}(I, X) = \{*\}$.
- *An object O in $\mathcal{C}$ is called* zero object *if it is both terminal and initial.*

We do not say exactly what, for example, a terminal object is (also because we have no idea of what objects are), we characterize it through its 'behaviour' in relation to other objects. This is the idea behind the formula '*[...] defined by a universal property*' that can be found in many texts. In this sense, we started a tradition that will reach its peak in Section 6.

Let us now see what Definition (3.6) means in the different categories we have already seen so far (before checking the table, I strongly recommend you to think about it yourself):

Category	Terminal	Initial	Zero
$\mathcal{S}$et	$\{*\}$	$\emptyset$	–
$\mathcal{G}$r	$<1>$	$<1>$	$<1>$
$\mathcal{A}b$	$<0>$	$<0>$	$<0>$
R-mod	$<0>$	$<0>$	$<0>$
$\mathcal{T}$op	$(\{*\},\{\{*\},\emptyset\})$	$(\emptyset,\{\emptyset\})$	–
$(S,\leq)$	$\max(S)$, if it exists	$\min(S)$, if it exists	–
$\mathcal{R}$ing	$<0>$	$\mathbb{Z}$	–
$\mathcal{R}$ng	$<0>$	$<0>$	$<0>$
$\mathcal{T}\text{op}_\bullet$	$(\{*\},\{\{*\},\emptyset\})$	$(\{*\},\{\{*\},\emptyset\})$	$(\{*\},\{\{*\},\emptyset\})$

Again, some comments might help, before those though, let's remark that the use of the article 'the' is justified by the Proposition (3.7). The different notation for $\mathcal{G}$r and $\mathcal{A}b$ is just motivated by the fact that for groups the multiplicative notation is used, while for abelian groups the additive one is used.

$\mathcal{S}$et The terminal object should be clear. For the initial object, remember that a function $f : A \to B$ is a relation $f \subset A \times B$ such that for every $a \in A$ [etc.]. Now, if A is the empty set, then the empty relation $\emptyset \subseteq \emptyset \times B = \emptyset$ is a function. And the only one possible. So we have our initial object in $\mathcal{S}$et.

From this, we can also conclude that there is no zero object in $\mathcal{S}$et (again, see Proposition (3.7)).

$\mathcal{T}$op This category is made of 'sets with some additional structure', that is a topology. Hence, it is somehow natural to go and pick the initial and terminal objects from $\mathcal{S}$et. Then, we need to put a topology over the singleton so that the unique map from a given topological space (X, τ_X) to the singleton is indeed a continuous map (otherwise, it would not appear as a morphism in our category $\mathcal{T}$op). Luckily, there is only one possible topology on the singleton, that is the trivial topology (that incidentally it is also the discrete topology). It is then a trivial exercise to verify that any function $f : X \to Y$ is continuous if we put the trivial topology on Y.

For the initial object as well, the same trick applies.

And then again, no zero object.

Proposition 3.7 *In a category $\mathcal{C}$,*

1. *All terminal objects, if any exists, are isomorphic up to a unique isomorphism;*
2. *All initial objects, if any exists, are isomorphic up to a unique isomorphism;*
3. *All zero objects, if any exists, are isomorphic up to a unique isomorphism.*
4. *A zero object exists if and only if both a terminal and an initial object exist and they are isomorphic.*

Proof: We will prove (1), we leave the others as an exercise.

Let T and S be terminal objects. By definition, $\text{Hom}_\mathcal{C}(T,S) \simeq \text{Hom}_\mathcal{C}(S,T) = \{*\}$ (if you wonder why '$\simeq$' and not '=', check Remarks (1.2)). Giving names, let's say $\text{Hom}_\mathcal{C}(T,S) = \{t\}$ and $\text{Hom}_\mathcal{C}(S,T) = \{s\}$. Now, clearly, also $\text{Hom}_\mathcal{C}(T,T) \simeq \text{Hom}_\mathcal{C}(S,S) = \{*\}$, by definition of terminal object. In this case, in particular, $\text{Hom}_\mathcal{C}(T,T) = \{\mathbb{1}_T\}$ and $\text{Hom}_\mathcal{C}(S,S) = \{\mathbb{1}_S\}$. Since also $t \circ s \in \text{Hom}_\mathcal{C}(S,S)$ and $s \circ t \in \text{Hom}_\mathcal{C}(T,T)$, we have part (1) of the proposition.

q.e.d.

Exercise 3.8 *Complete the proof of Proposition (3.7).*

4 Natural transformations and equivalence of categories

4.1 Natural transformations

So far, we have defined what a category is. Then, we have made them a category (of categories) with arrows the functors. Now we want to make the latters too into a category, so we need arrows between functors.

Definition 4.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and F and G two functors from $\mathcal{C}$ to $\mathcal{D}$. A* natural transformation φ from F to G *is a family $\{\varphi_X\}_{X\in\mathrm{Obj}(\mathcal{C})}$, indexed by the objects of the category $\mathcal{C}$, of morphisms $\varphi_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ in the category $\mathcal{D}$ satisfying the following condition:*
for every morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\varphi_Y]{} & G(Y) \end{array} .$$

commutes. In formulae: for every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have $\varphi_Y \circ F(f) = G(f) \circ \varphi_X$.

Remark 4.2

Ok ... but why?!

This definition is as important as not clear at the beginning (I think). In the end, obviously it is your job to understand it, but let's see it this way: a functor F is a way to translate objects and morphisms *from a category $\mathcal{C}$ to a category $\mathcal{D}$. Having two functors, that is having also a G, means having two such translations. A natural transformation is a way to relate these two different translations. Hence, for each of the objects X (in $\mathcal{C}$), we want a morphism (in $\mathcal{D}$) between the two images, $F(X)$ and $G(X)$, of X. To this aim, here is the family $\{\varphi_X\}_{X\in\mathrm{Obj}(\mathcal{C})}$. This is the easy part. Being all based on morphisms, this family must behave well with the other relevant morphisms (once again, the motto is "morphisms are what counts"). Given then a morphism in $\mathcal{C}$, that is an $f : X \to Y$, we can apply either F or G to it. The two φ_X and φ_Y should then be 'compatible' with $F(f)$ and $G(f)$. The precise meaning of 'compatible' is the diagram in the definition above.*

Let's try to show this in an example:

Example 4.3 *Let $\mathcal{C}$ be the category Vect_k^f of finitely generated vector spaces over a field k as objects and linear functions as morphisms.*

As $\mathcal{D}$ we take $\mathrm{Obj}(\mathcal{D}) = \mathbb{N}$ with morphisms

$$\mathrm{Hom}_{\mathcal{D}}(m, n) = \mathrm{Mat}_k(m, n) = \{m \textit{ rows } n \textit{ columns matrices with entries in } k\}.$$

Instead of $\mathcal{D}$, we will write directly $\mathbb{N}$ and $\mathrm{Hom}_{\mathbb{N}}$ throughout this example.

Now we need to define $F : \mathrm{Vect}_k^f \to \mathbb{N}$: given a finitely generated k-vector space V, we want to associate to it a natural number. I guess an unsurprising choice is $F(V) = \dim_k(V)$. Once again, the core part should be defining F on the arrows. So let's take a linear function $f : V \to W$ and we need to define $F(f) : m \to n$, where $m = \dim_k(V)$ and $n = \dim_k(W)$. From the definition of $\mathbb{N}$, we need to pick a $m \times n$ matrix, starting from the linear function f. Again unsurprisingly, we choose the representative matrix of f. Of course, before that, we need bases! So before *defining F, we have to fix a basis for* each *vector space V in Vect_k^f (and so, incidentally, we need the Axiom of Choice). Once this is done, we have F.*

What happens now if we make a different choice for the bases? On the objects, nothing would change, but on the arrows of course it would; hence, we would get a different functor, let's call it G. To sum up, for each different choice of a system of bases, one basis for each vector space, we have a different functor. The functors F and G are two of these.

Two such F and G are clearly identical on the objects, they differ on the morphisms. But how do they differ? How can we relate them? Here comes a natural transformation. In this case, such a φ would be a family $\varphi_V : m \to m$ of matrices $m \times m$, where $m = \dim_k(V)$. Remembering (previous paragraph) that each functor comes equipped with a basis for V, we can take as φ_V the matrix that represents the base change, let's denote it then by M_V (but in fact, we could just use φ_V). The identity $\varphi_Y \circ F(f) = G(f) \circ \varphi_X$ defining a natural transformation translates now into $M_W \cdot F(f) = G(f) \cdot M_V$, that is $F(f) = M_W^{-1} G(f) M_V$. As we all learnt in our first course of Linear Algebra, this is true.

Before proceeding, let me collect here a few definitions of important types of functors:

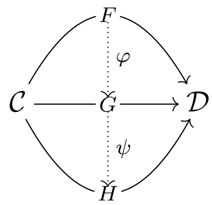
Definition 4.4 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

- *F is* full *if for every X and Y in $\mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective.*
- *F is* faithful *if for every X and Y in $\mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective.*
- *F is* fully faithful *if it is both full and faithful.*

Exercise 4.5 *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Show that, if the image $F(f)$ of a morphism f in $\mathcal{C}$ is an isomorphism, also f is an isomorphism. This completes Exercise (2.11).*

Example 4.6 *Again from Linear Algebra, we know that any of the functors defined in Example (4.3) is fully faithful.*

Natural transformations are called like this for historical reasons, but they should be called morphisms of functors. Before that though, one should remark that they do, actually, satisfy the axioms of the morphisms of a category. To do that, we need to define the composition: given a diagram like



,

that is three functors F, G, and H from a category $\mathcal{C}$ to a category $\mathcal{D}$, together with natural transformations $\varphi = \{\varphi_X\}_{X \in \mathrm{Obj}(\mathcal{C})} : F \to G$ and $\psi = \{\psi_X\}_{X \in \mathrm{Obj}(\mathcal{C})} : G \to H$, we define

$$\psi \circ \varphi = \{\psi_X \circ \varphi_X\}_{X \in \mathrm{Obj}(\mathcal{C})} : F \to H.$$

This is indeed a natural transformation because of the commutative diagram below:

$$\begin{array}{ccccc} F(X) & \xrightarrow{\varphi_X} & G(X) & \xrightarrow{\psi_X} & H(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\ F(Y) & \xrightarrow[\varphi_Y]{} & G(Y) & \xrightarrow[\psi_Y]{} & H(Y) \end{array} .$$

Since also $\mathbb{1}_F = \{\mathbb{1}_{F(X)}\}_{X \in \mathcal{C}}$ is a morphism of functors and composition is associative, we can finally give the following:

Definition 4.7 *Given two categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ the category with objects the functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms the natural transformations.*

Exercise 4.8 *Show that a morphism $\varphi = \{\varphi_X\}$ in $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$ is an isomorphism if and only if each φ_X is an isomorphism.*

Exercise 4.9 *Show that, given a group G, the category of $G-$modules is the same as the category $\mathrm{Fct}(G, \mathcal{A}b)$, regarding G as a category with a single object as usual.*

The moment has come to add one more piece to the discussion started with Example (2.3.5). We observed there that from any fixed object X in a category $\mathcal{C}$ we can construct the two functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X)$. Assume now we have a second object X' in $\mathcal{C}$, then we want to show that

Proposition 4.10 *For every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ in a category $\mathcal{C}$, we have morphisms of functors $\varphi^f : h_{\mathcal{C}}(X) \to h_{\mathcal{C}}(X')$ and $\psi^f : k_{\mathcal{C}}(X) \leftarrow k_{\mathcal{C}}(X')$ (mind the arrow).*

Proof: We will work out this last one and leave the other as an exercise.

Once again, we need to define a family of morphisms $\psi^f = \{\psi^f_Y\}$ with $\psi^f_Y : k_{\mathcal{C}}(X')(Y) \to k_{\mathcal{C}}(X)(Y)$, that is $\psi^f_Y : \mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$. An unsurprising candidate is to use $\psi^f_Y = f_* : \alpha \mapsto \alpha \circ f$, where α is a generic morphism in $\mathrm{Hom}_{\mathcal{C}}(X', Y)$. The diagram whose commutativity we need to prove is then, given a morphism $g : Y \to Y'$, the following:

$$\begin{array}{ccc} k_{\mathcal{C}}(X')(Y) & \xrightarrow{\psi^f_Y} & k_{\mathcal{C}}(X)(Y) \\ {\scriptstyle k_{\mathcal{C}}(X')(g)}\downarrow & & \downarrow{\scriptstyle k_{\mathcal{C}}(X)(g)} \\ k_{\mathcal{C}}(X')(Y') & \xrightarrow[\psi^f_{Y'}]{} & k_{\mathcal{C}}(X)(Y') \end{array} .$$

Applying the definitions we get:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X', Y) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ {\scriptstyle g^\star}\downarrow & & \downarrow{\scriptstyle g^\star} \\ \mathrm{Hom}_{\mathcal{C}}(X', Y') & \xrightarrow[f_*]{} & \mathrm{Hom}_{\mathcal{C}}(X, Y') \end{array} ;$$

taking now a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$, we obtain

$$\begin{array}{ccc} \alpha & \xmapsto{\quad f_* \quad} & \alpha \circ f \\ {\scriptstyle g^\star}\downarrow & & \downarrow{\scriptstyle g^\star} \\ g \circ \alpha & \xmapsto[f_*]{} & (g \circ \alpha) \circ f = g \circ (\alpha \circ f) \end{array} ,$$

that is true, by the axiom of composition in Definition (1.1.3b).

q.e.d.

Exercise 4.11 *Complete the proof of Proposition (4.10) for* φ^f*.*

4.2 Equivalences of categories

We have defined long ago (see Definition (1.6)) what an isomorphism is. Someone may then wonder why we have not talked about isomorphisms of categories. Of course we could, but they are not particularly interesting, because isomorphic categories are really too similar. The reason we are mentioning this now, is because we are ready to introduce a more interesting concept:

Definition 4.12 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is said to be* an equivalence of categories *if there exists another functor* $G : \mathcal{D} \to \mathcal{C}$ *such that*

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}} \quad \text{in } \mathrm{Fct}(\mathcal{C}, \mathcal{C}) \qquad \text{and} \qquad F \circ G \simeq \mathbb{1}_{\mathcal{D}} \quad \text{in } \mathrm{Fct}(\mathcal{D}, \mathcal{D}).$$

We then say that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent *and we call* G *a* quasi-inverse of F.

Example 4.13 *We continue with Example (4.3) and show that in fact the functors* $F : \mathrm{Vect}^f_k \to \mathbb{N}$ *(each depending on the choice of a collection of bases, one for each vector space* V*) are equivalences of categories.*

In order to do that, we need to define a functor $G : \mathbb{N} \to \mathrm{Vect}^f_k$ *and prove that it is a quasi-inverse of* F*. Given an object* $m \in \mathbb{N}$*, not surprisingly we assign* $G(m) = k^m$*. Given a morphism* $f : m \to n$*, that is a matrix* M *in* $\mathrm{Mat}_k(m, n)$*, we define* $G(f)$ *to be the linear map defined by* M *choosing the canonical bases for* k^m *and* k^n*.*

By Definition (4.12), we now need two isomorphisms of functors $\varphi : G \circ F \xrightarrow{\sim} \mathbb{1}_{\mathrm{Vect}^f_k}$ *and* $\psi : F \circ G \xrightarrow{\sim} \mathbb{1}_{\mathbb{N}}$*. We start with* φ*, step by step:*

- *for each object* $V \in \mathrm{Vect}^f_k$*, we need to define* φ_V*. This has to be a morphism in* Vect^f_k*, that is a linear map from* $G \circ F(V) = k^{\dim(V)}$ *to* $\mathbb{1}_{\mathrm{Vect}^f_k}(V) = V$*;*

- *remember that, coming together with F, we have a choice of a basis $\{v_i\}_{i=1}^{\dim(V)}$ for each V, so we can define φ_V as the linear map such that $\varphi_V(e_i) = v_i$;*
- *in order to verify that the collection $\varphi = \{\varphi_V\}$ is actually a morphism of functors, we need to take a morphism of vector spaces $f : V \to W$ in Vect_k^f and check that the following diagram commutes:*

$$\begin{array}{ccccccc} G\circ F(V) & = & k^{\dim(V)} & \xrightarrow{\varphi_V} & V & = & \mathbb{1}_{\mathrm{Vect}_k^f}(V) \\ & & \downarrow{\scriptstyle G\circ F(f)} & & \downarrow{\scriptstyle f} & & \\ G\circ F(W) & = & k^{\dim(W)} & \xrightarrow[\varphi_W]{} & W & = & \mathbb{1}_{\mathrm{Vect}_k^f}(W) \end{array}.$$

Convince yourself that this is true, by construction.

Now it is the turn of ψ, so we have to define ψ_m's such that

$$\begin{array}{ccccccc} F\circ G(m) & = & m & \xrightarrow{\psi_m} & m & = & \mathbb{1}_{\mathbb{N}}(m) \\ & & \downarrow{\scriptstyle G\circ F(M)} & & \downarrow{\scriptstyle M} & & \\ F\circ G(n) & = & n & \xrightarrow[\psi_n]{} & n & = & \mathbb{1}_{\mathbb{N}}(n) \end{array},$$

for any $m \times n$ matrix M. This is easy but not as trivial as it looks like, since $G \circ F(M)$ is not *necessarily M itself. Remember that $F(M)$ is the linear map $k^m \to k^n$ represented by M* with respect to the the fixed basis related to F, *but there is* no reason *why they should be the canonical bases. When they are, the ψ_m's are the $m \times m$ identity matrix, but normally they have to be the base change matrices from the canonical basis to the basis attached to F.*

Exercise 4.14 *What if, when defining G, we decided to use the bases given by F, rather than the canonical bases?*

Proving that two categories are equivalent usually requires deep results on their structures. Hence, they are not trivial statements. We can list here a few of them, with the understanding that, on average, a course each is usually required to prove them.

Example 4.15

- *The category of compact topological abelian groups $\mathcal{CA}b$ is equivalent to $\mathcal{A}b^{\mathrm{op}}$, by the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{R}/\mathbb{Z})$. This is called Pontryagin Duality.*
- *The category of commutative Banach algebras with an involution is equivalent to the category of compact Hausdorff topological spaces.*
- *The category of affine schemes is equivalent to $\mathcal{CR}\mathrm{ing}^{\mathrm{op}}$, category of commutative rings with identity.*

Definition 4.16 *A category that is equivalent to its opposite category is called* self-dual.

Example 4.17

1. *The category Vect_k^f is self-dual. An equivalence is given by $\mathrm{Hom}_k(-, k)$.*
2. *The category $\mathcal{A}b^f$ of finite abelian groups is self-dual. An equivalence is given by $\mathrm{Hom}_Z(-, \mathbb{Q}/\mathbb{Z})$.*
3. *The category $\mathcal{S}\mathrm{et}$ is not self-dual:*

   Exercise 4.18 *Prove this, comparing its terminal and initial objects.*

Exercise 4.19 *Prove that, if F is an equivalence of categories, then F is fully faithful.*

Definition 4.20 *Given a category $\mathcal{C}$, a* subcategory of $\mathcal{C}$ *is a category $\mathcal{C}'$ such that*

1. $\mathrm{Obj}(\mathcal{C}') \subseteq \mathrm{Obj}(\mathcal{C})$;

2. *For every X, Y in* $\mathrm{Obj}(\mathcal{C}')$, $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$*;*

3. *For every X, Y, Z in* $\mathrm{Obj}(\mathcal{C}')$*, the composition* $\circ^{\mathcal{C}'}_{XYZ}$ *is the restriction of* $\circ^{\mathcal{C}}_{XYZ}$*.*

Recalling Definition (4.4), it follows that to each subcategory of a category one can associate a faithful functor ι and, viceversa, that to each faithful functor F corresponds a subcategory, that is its image. We may willingly confuse the two concepts.

Definition 4.21

1. *A subcategory* $\mathcal{C}' \xrightarrow{\iota} \mathcal{C}$ *is* full *if ι is full.*

2. *A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is* dense *if for every object X of $\mathcal{C}$ there exists an object X' of $\mathcal{C}'$ such that $X \simeq X'$.*

Time for examples:

Example 4.22

1. *The category $\mathcal{A}b$ is a full subcategory of* $\mathcal{G}\mathrm{r}$.

2. *The category* $\mathcal{M}\mathrm{etr}$ *of metric spaces with metric functions as morphisms (i.e. given (X, d_X) and (Y, d_Y), functions $f : X \to Y$ such that $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$) is a non-full subcategory of* $\mathcal{T}\mathrm{op}$.

3. *Given a field k, the category* $\mathcal{F}\mathrm{ld}_k$ *of k-field extensions, with field homomorpshims leaving fixed the base field k as morphisms, is a non-full subcategory of* Vect_k*.*

4. *The category $\mathbb{N}$ described in Example (4.3) and Example (4.13) is, via the functor G, a full subcategory of* Vect^f_k*.*

The last example above can be generalized:

Theorem 4.23 *Assume that $\mathcal{C}'$ is a full and dense subcategory of a category $\mathcal{C}$. Then $\mathcal{C}'$ is equivalent to $\mathcal{C}$.*

Exercise 4.24 *Prove the theorem above.*

Definition 4.25 *A subcategory $\mathcal{C}'$ as in Theorem (4.23) and such that $X' \simeq Y'$ if and only if $X' = Y'$, is called a* skeleton *of $\mathcal{C}$.*

Exercise 4.26

1. *Every category $\mathcal{C}$ has a skeleton (assuming the Axiom of Choice).* (Hint: being isomorphic is an equivalence relation.)

2. *All skeletons of a category $\mathcal{C}$ are equivalent.*

Exercise 4.27 *The category $\mathbb{N}$, as above, is a skeleton of* Vect^f_k*.*

5 Yoneda Embedding and Yoneda Lemma

We now go back to the definition of a category and we give a deeper look to the sets of morphisms. In Example (2.3.5) and then again in Proposition (4.10), we have already noticed that, once we fix an object X in a category $\mathcal{C}$, we obtain two functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X)$. A lot in Mathematics depends on them, so let's recall the whole definition.

Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$, we have $h_{\mathcal{C}}(X)$ and $k_{\mathcal{C}}(X)$ defined as follows:

On the objects | *On the morphisms*

$$h_{\mathcal{C}}(X) : \begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{S}\mathrm{et} \\ Y & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(Y, X) \end{array} \qquad \begin{array}{c} Y \\ \downarrow f \\ Y' \end{array} \longmapsto \begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(Y', X) \\ \downarrow f_* \\ \mathrm{Hom}_{\mathcal{C}}(Y, X) \end{array} \quad \begin{array}{c} \alpha \\ \downarrow \\ \alpha \circ f \end{array}$$

$$k_{\mathcal{C}}(X) : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{S}\mathrm{et} \\ Y & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(X, Y) \end{array} \qquad \begin{array}{c} Y \\ \downarrow g \\ Y' \end{array} \longmapsto \begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ \downarrow g^* \\ \mathrm{Hom}_{\mathcal{C}}(X, Y') \end{array} \quad \begin{array}{c} \alpha \\ \downarrow \\ g \circ \alpha \end{array}$$

To have our notation more consistent throughout our discussion, we make two notation adjustments and we obtain:

$$
\begin{array}{cccc}
 & & \textit{On the objects} & \\
h_{\mathcal{C}}(Y): & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{S}\mathrm{et} \\
 & X & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(X,Y)
\end{array}
\qquad
\begin{array}{ccccc}
 & & \textit{On the morphisms} & & \\
X & & \mathrm{Hom}_{\mathcal{C}}(X,Y) & \alpha & \\
\downarrow f & \longmapsto & \downarrow f_* & \downarrow & \\
X' & & \mathrm{Hom}_{\mathcal{C}}(X',Y) & \alpha\circ f & .
\end{array}
$$

$$
\begin{array}{cccc}
k_{\mathcal{C}}(X): & \mathcal{C} & \longrightarrow & \mathcal{S}\mathrm{et} \\
 & Y & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(X,Y)
\end{array}
\qquad
\begin{array}{cccc}
Y & & \mathrm{Hom}_{\mathcal{C}}(X,Y) & \alpha \\
\downarrow g & \longmapsto & \downarrow g^* & \downarrow \\
Y' & & \mathrm{Hom}_{\mathcal{C}}(X,Y') & g\circ\alpha
\end{array}
$$

Please note that the morphism $X \xrightarrow{f} X'$ is from $\mathcal{C}^{\mathrm{op}}$, that means that we have a morphism $X \xleftarrow{f} X'$ in $\mathcal{C}$, so the composition $\alpha \circ^{\mathcal{C}}_{X'XY} f$, performed in $\mathcal{C}$, does make sense.

Looking at what we have written, all the the diagrams we have given actually come from a single one:

$$
X \xleftarrow{f} X' \quad Y \xrightarrow{g} Y' \quad
\begin{array}{ccc}
\alpha & \longmapsto & g\circ\alpha \\
\Big\downarrow & \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{g^*} & \mathrm{Hom}_{\mathcal{C}}(X,Y') \\ f_*\downarrow & & \downarrow f_* \\ \mathrm{Hom}_{\mathcal{C}}(X',Y) & \xrightarrow[g^*]{} & \mathrm{Hom}_{\mathcal{C}}(X',Y') \end{array} & \Big\downarrow \\
\alpha\circ f & \longmapsto & g\circ\alpha\circ f
\end{array} .
$$

As we remarked already when discussing about morphisms of functors, this is actually just a diagram representing the associativity of composition of morphisms. Hence, once again by Definition (1.1), by saying this we mean that even if this discussion is deep, it is technically basic.

In conclusion, using the notation

$$\hat{\mathcal{C}} = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \qquad \text{and} \qquad \check{\mathcal{C}} = \mathrm{Fct}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}},$$

we have two functors

$$h_{\mathcal{C}} : \mathcal{C} \to \hat{\mathcal{C}} \qquad k_{\mathcal{C}} : \mathcal{C} \to \check{\mathcal{C}}.$$

A few remarks now, and some later:

Various remarks 5.1

1. *Yes, it is* $\check{\mathcal{C}} = \mathrm{Fct}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}^{\mathrm{op}})$. *If not, it would have to be* $k_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fct}(\mathcal{C}, \mathcal{S}\mathrm{et})$ *and other arrows would be unhappy. Try and convince yourself (if you really want to spend extra time with diagrams).*

2. *Why on Earth..?! Well, you can look at* $h_{\mathcal{C}}(Y)$ *(or* $k_{\mathcal{C}}(X)$*) as a (*the*) way to encode* the entire collection of morphisms to Y *(or from* X*) in a single entity; this entity is an object of the category* $\mathrm{Fct}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ *(or* $\mathrm{Fct}(\mathcal{C}, \mathcal{S}\mathrm{et})^{\mathrm{op}}$*). Considering the motto "morphisms are what count", it should not come as a surprise the intention of studying* $h_{\mathcal{C}}(Y)$ *instead of* Y*. The precise meaning of this statement is a bit below, but one of the natural questions can be anticipated: if we know that two objects* X_1 *and* X_2 *are such that the morphisms from them to* any *other object* Y *are always the same, that is if we have* $\mathrm{Hom}_{\mathcal{C}}(X_1, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X_2, Y)$ *for every* Y *in a compatible way, can we conclude that* $X_1 \simeq X_2$*? If all the plan to concentrate on morphisms is meant to make any sense, the answer must be yes (with one extra requirement).*

3. *In countless many situations in geometry and physics we have dualities or situations where arrows get inverted. They pretty much all come from the fact that* Hom *has one contravariant component. And there is nothing we can do about it.*

Ok, so in practice we send $\mathcal{C}$ in the two categories $\hat{\mathcal{C}}$ and $\check{\mathcal{C}}$. What is happening to it? The answer is much stronger than the question:

Theorem 5.2 (Yoneda Lemma) *If* $\mathcal{C}$ *is a category,* X *an object of* $\mathcal{C}$*, and* $A \in \hat{\mathcal{C}}$ *and* $B \in \check{\mathcal{C}}$*, then*

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\mathcal{C}}(X), A) = A(X) \qquad \textit{and} \qquad \mathrm{Hom}_{\check{\mathcal{C}}}(B, k_{\mathcal{C}}(X)) = B(X).$$

Proof: Exercise! Or see any book of category theory.

q.e.d.

Corollary 5.2.1 *The functors $h_{\mathcal{C}}$ and $k_{\mathcal{C}}$ are fully faithful (see Definition (4.4)).*

Proof: Indeed, taking $A = h_{\mathcal{C}}(Y)$ and $B = k_{\mathcal{C}}(X)$, we obtain, from Theorem (5.2),

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\mathcal{C}}(X), h_{\mathcal{C}}(Y)) = A(X) = h_{\mathcal{C}}(Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

$$\mathrm{Hom}_{\check{\mathcal{C}}}(k_{\mathcal{C}}(X), k_{\mathcal{C}}(Y)) = B(X) = k_{\mathcal{C}}(X)(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

q.e.d.

Corollary 5.2.2 *Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(W, Y)$ (or $\mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, W)$) is an isomorphism for every W, then f is also an isomorphism.*

Proof: Indeed, '$f\circ$' is $h_{\mathcal{C}}(f)$, and since the latter is fully faithful, f needs to be an isomorphism (see Exercise (4.5)). The same is for '$\circ f$' and $k_{\mathcal{C}}(f)$.

q.e.d.

More remarks!

Various remarks 5.3

1. *Corollary (5.2.1) means that the two functors $h_{\mathcal{C}}$ and $k_{\mathcal{C}}$ embed the category $\mathcal{C}$ into $\hat{\mathcal{C}}$ and $\check{\mathcal{C}}$, respectively.*
2. *In other words, that we can willingly confuse X and $h_{\mathcal{C}}(X)$, to the point that it is a common notation to write $X(Y)$ for $\mathrm{Hom}_{\mathcal{C}}(Y, X)$, that is viewing the object X as if it was the functor $h_{\mathcal{C}}(X)$.*
3. *In other words, working with the object X or the object $h_{\mathcal{C}}(X)$ is the exact same thing, indeed the collection of morphisms to Y (or from X) does determine Y (or X), as anticipated in Remark (5.1.2) above and proved by Corollary (5.2.2). The extra requirement is the existence of at least one morphism from X to Y.*

Definition 5.4 *Given a category $\mathcal{C}$, a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ (or $\mathcal{C} \to \mathcal{S}\mathrm{et}$) is* representable *if there exists an object X in $\mathcal{C}$ such that $F \simeq h_{\mathcal{C}}(X)$ (or $F \simeq k_{\mathcal{C}}(X)$). The object X is called* a representative of F.

Thanks to Yoneda Lemma (more precisely, by Corollary (5.2.2)), two representatives of a representable functor isomorphic via a unique isomorphism.

Exercise 5.5 *Let $\mathcal{C}$ be a category. Show that the constant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ defined by*

$$F(X) = \{*\} \qquad \text{for every } X,$$

$$F(f) = \mathbb{1}_{\{*\}} \qquad \text{for every } f,$$

is representable if and only if $\mathcal{C}$ has a terminal object.

Exercise 5.6 *Prove that the forgetful functor $\rho : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ is representable.*

Example 5.7 *If you have encountered it already, the tensor product of modules has yet one more way to be defined: let R be a ring, A a right $R-$module, and B a left $R-$module. The* tensor product $A \otimes_R B$ of A and B over R *is the abelian group representing the functor*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, -) : R\text{–mod} \to \mathcal{A}b\ .$$

Here $\mathrm{Hom}_R(B, -)$ is given a structure of letf R-module as

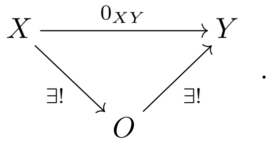
$$rf : b \mapsto f(br)\ .$$

6 Kernels, products, and stuff

We now go back to work inside a fixed category $\mathcal{C}$. We continue the game of redefining known concepts but without mentioning elements, and using the new terminologies and tools from the previous section.

6.1 Kernels, cokernels, images, and coimages

Definition 6.1 *Let $\mathcal{C}$ be a category and assume that there exists a zero object O (see Definition (3.6)). Given any pair of objects X, Y in $\mathcal{C}$, we call* zero morphism, denoted by 0_{XY}, *the unique morphism $X \to Y$ that factors through O. In a diagram:*



The uniqueness comes directly by the definition of zero object. We will often omit the index for the zero morphism, unless a confusion might arise.

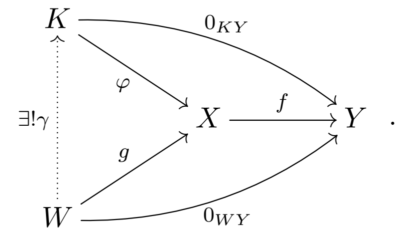
Exercise 6.2 *Prove that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we have $f \circ 0_{WX} = 0_{WY}$ and $0_{YZ} \circ f = 0_{XZ}$.*

Definition 6.3 *Let $\mathcal{C}$ be a category and assume that there exists a zero object O in $\mathcal{C}$. Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we say that a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(K, X)$ is a* kernel of f *if*

1. *$f \circ \varphi = 0_{KY}$;*
2. *For every other morphism $g \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ such that $f \circ g = 0_{WY}$, there exists a unique morphism $\gamma \in \mathrm{Hom}_{\mathcal{C}}(W, K)$ such that $g = \varphi \circ \gamma$.*

Various remarks 6.4

1. *The definition just given should be more clear by representing it as a commutative diagram:*

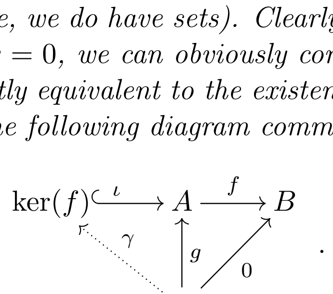


2. *Another way to see it is the following: looking at φ and g, we can see that they behave, with respect to f, in the same way: they give the zero morphism if precomposed with f. What makes φ special, is that it is in some sense minimal with respect to this property. And the "some sense" is exactly that every other morphism with the same property, g being one of them, factors through it in a unique way. Hence the existence of a unique γ.*
3. *This technique to define objects is normally referred by "definition by a universal property", in the sense that we define an object or a morphism by describing its behaviour with respect to other entities. With have already used the same technique when we defined terminal/initial/zero objects. This technique ensure uniqueness (up to a unique isomorphism), but clearly not existence. For the latter, extra criteria are necessary and usually they are found case by case.*

Example 6.5 *We consider the category $\mathcal{A}b$, but $R-mod$ works the same for any ring R. Given a group homomorphism $f : A \to B$, and using the usual old standard notation, we can write the diagram*

$$\ker(f) \xhookrightarrow{\iota} A \xrightarrow{f} B \,,$$

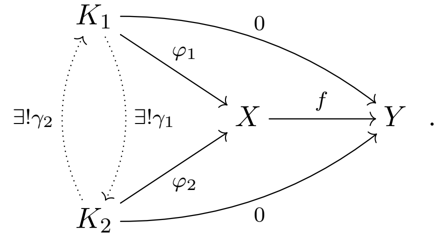
where ι is the standard inclusion (this time, we do have sets). Clearly $f \circ \iota = 0$. If now we have another group homomorphism $g : C \to A$ such that $f \circ g = 0$, we can obviously conclude that $\mathrm{im}(g) \subset \ker(f)$ (again, we do have sets). But this last statement is exactly equivalent to the existence of a group homomorphism, that by no coincidence we denote by γ, that makes the following diagram commute:



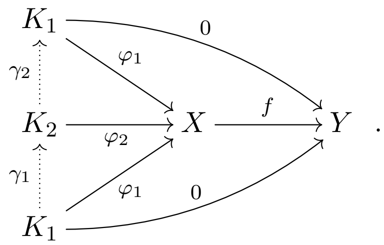
Since ι is injective, γ is unique. In conclusion, $\mathcal{A}b$ does have kernels, with the difference that we decided to call kernels the morphisms, while normally we refer to the subgroups. On the other hand, it should be clear by now what we care the most between objects and morphisms...

Proposition 6.6 *Let $\mathcal{C}$ be a category with a zero object O and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If $\varphi_1 : K_1 \to X$ and $\varphi_2 : K_2 \to X$ are two kernels of f, then there exists a unique isomorphism $\alpha : K_1 \to K_2$ such that $\varphi_1 = \varphi_2 \circ \alpha$.*

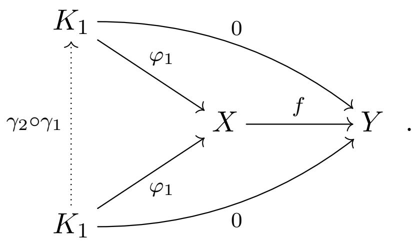
Proof: If we write down the diagram for φ_1 and φ_2, we obtain



We can somehow expand this diagram into



Since every triangle is commutative in the above diagram, we can remove K_2 and obtain



By definition, there can be only one morphism like $\gamma_2 \circ \gamma_1$ making this diagram commutative, and since obviously also $\mathbb{1}_{K_1}$ would do, we conclude that indeed $\gamma_2 \circ \gamma_1 = \mathbb{1}_{K_1}$. Taking as α the morphism γ_1, and doing the same work with $\gamma_1 \circ \gamma_2$, we have the proposition.

q.e.d.

It might be worth it to remark also the following:

Proposition 6.7 *Let $\mathcal{C}$ be a category with a zero object O and let f be a morphism in $\mathcal{C}$. Any kernel of f is a monomorphism*

Exercise 6.8 *Prove Proposition (6.7).*

The above proposition allows us to see a kernel also as a subobject of the domain of a morphism. We can now denote by $\ker(f)$ this subobject and restore the intuition from standard algebra.

Example 6.9 *Let's make an example out of Example (6.5). Take the groups $\mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, as in Example (3.3), and the group homomorphism $1 \mapsto 1$ from $\mathbb{Z}/6\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. We then have two kernels, in the categorical sense:*

$$\varphi_1 : \begin{array}{ccc} \mathbb{Z}/3\mathbb{Z} & \to & \mathbb{Z}/6\mathbb{Z} \\ 1 & \mapsto & 2 \end{array}, \qquad \varphi_2 : \begin{array}{ccc} \mathbb{Z}/3\mathbb{Z} & \to & \mathbb{Z}/6\mathbb{Z} \\ 1 & \mapsto & 4 \end{array},$$

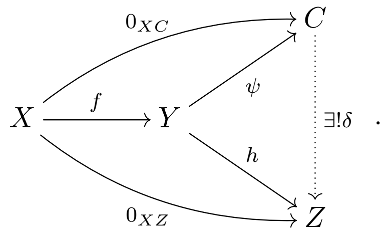
since the kernel, in the old sense, is clearly $\{0, 2, 4\}$. By Proposition (6.6), there must be an isomorphism α, and this is clearly given by $\alpha : 1 \mapsto 2$. Do check this.

Dual to the notion of kernel, is the notion of cokernel.

Definition 6.10 *Let $\mathcal{C}$ be a category and assume that there exists a zero object O in $\mathcal{C}$. Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we say that a morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(Y,C)$ is a* cokernel of f *if*

1. $\psi \circ f = 0_{XC}$;
2. *For every other morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ such that $h \circ f = 0_{XZ}$, there exists a unique morphism $\delta \in \mathrm{Hom}_{\mathcal{C}}(C,Z)$ such that $h = \delta \circ \psi$.*

The corresponding diagram then this time is



Exactly as we have done for monomorphisms and epimorphisms (see Proposition (1.17), we can prove that

Proposition 6.11 *Let $\mathcal{C}$ be a category and assume that there exists a zero object O in $\mathcal{C}$. Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, a morphism ω is a kernel for f in $\mathcal{C}$ if and only if it is a cokernel for f in $\mathcal{C}^{\mathrm{op}}$.*

Exercise 6.12 *Prove Proposition (6.11).*

We could have actually given this as a definition and derive the previous diagram from the one of kernels. In any case, whatever we will prove for kernels from now on, will also be valid for cokernels, provided that we invert all the arrows. For example, as corollary of Proposition (6.7), we have

Proposition 6.13 *Let $\mathcal{C}$ be a category with a zero object O and let f be a morphism in $\mathcal{C}$. Any cokernel of f is an epimorphism.*

Proof: As we said, it is a corollary of Proposition (6.7) and Proposition (6.11).

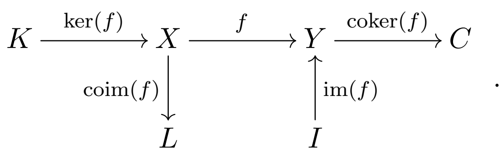
q.e.d.

We can briefly continue our discussion including two more definitions:

Definition 6.14 *Let $\mathcal{C}$ be a category with a zero object O and let f be a morphism in $\mathcal{C}$.*

- *An* image of f *is, if it exists, a kernel of a cokernel of f;*
- *A* coimage of f *is, if it exists, a cokernel of a kernel of f;*

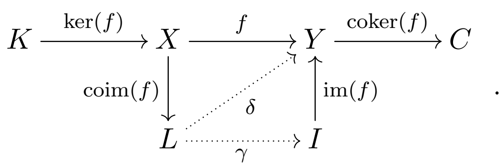
With an abuse of notation, since none of these morphisms is unique, but only almost unique, let us denote them by $\ker(f)$, $\mathrm{coker}(f)$, $\mathrm{im}(f)$, and $\mathrm{coim}(f)$. We can now put all of them in one diagram that looks like this:



Now, by definition of kernel, we have $f \circ \ker(f) = 0$, and by definition of cokernel (of the kernel), there must be a unique $\delta : L \to Y$ such that $f = \delta \circ \mathrm{coim}(f)$. Now, from the fact that $\mathrm{coim}(f)$ is a cokernel, and hence an epimorphism, and using Exercise (6.2) together with the equalities

$$0 = \mathrm{coker}(f) \circ f = (\mathrm{coker}(f) \circ \delta) \circ \mathrm{coim}(f) = 0 \circ \mathrm{coim}(f) = 0,$$

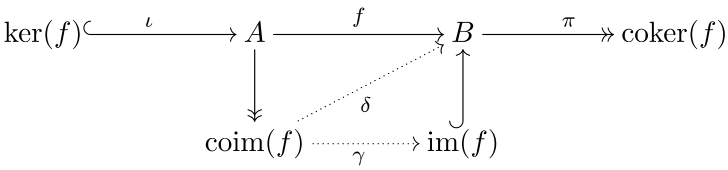
we obtain that also $\mathrm{coker}(f) \circ \delta = 0$. Then by definition of kernel (of the cokernel), there must be a unique $\gamma : L \to I$ such that $\delta = \mathrm{im}(f) \circ \gamma$. Filling up the diagram above, we have obtained the following commutative diagram:



Definition 6.15 *Let $\mathcal{C}$ be a category with a zero object O and let f be a morphism in $\mathcal{C}$. If the relevant kernels and cokernels exists, we say that f* is a proper morphism *if γ is an isomorphism.*

Let us see this in an example:

Example 6.16 *Let us take the category $\mathcal{A}b$. In this category, every morphism has kernels and cokernels in the categorical sense. So it also have images and coimages. Willingly mixing up the notations between category and algebra, we have the diagram:*



.

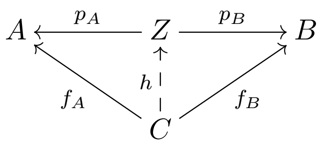
Rephrasing the concepts we have defined above, now in common algebraic terms, we obtain the equalities $\mathrm{coker}(f) = B/\mathrm{im}(f)$ *and* $\mathrm{coim}(f) = A/\ker(f)$. *But then, saying that f is proper reduces to the isomorphism* $A/\ker(f) \simeq \mathrm{im}(f)$. *And we all studied this in our first algebra course.*

Example 6.17 *Let G be a non trivial abelian group. We can see it as a topological group either with the discrete topology or the trivial topology. The identity map, with the discrete topology as domain topology and the trivial topology as codomain topology, is a non proper morphism.*

The discussion should now continue introducing abelian categories, where every morphism being proper is one of the axioms. Abelian categories, modeled on $\mathcal{A}b$ or in general on $R-$mod, are a milestone in category theory, due largely to Grothendieck. We have to direct the reader to any of the references indicated.

6.2 Products and sums

Definition 6.18 *Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Obj}(\mathcal{C})$.* A product of A and B in $\mathcal{C}$ *is a triple (Z, p_A, p_B) such that $p_A : Z \to A$ and $p_B : Z \to B$ are morphisms in $\mathcal{C}$ with the property that, for every other object C and morphisms $f_A : C \to A$ and $f_B : C \to B$, there exists a unique morphism $h : C \to Z$ such that $p_A \circ h = f_A$ and $p_B \circ h = f_B$. In diagrams, there is a unique h that makes the following diagram commutative:*



.

As an example, realize that in $\mathcal{S}$et a/the product of two sets is their Cartesian product, together with the two projection.

Proposition 6.19 *Products, when they exist, are unique up to a unique isomorphism.*

Proof: There are two possible versions of such a proof:
Version 1:

Exercise 6.20 *Prove the proposition playing with the diagrams, as in the proof of Proposition (6.6).*

Version 2: Define a functor F as follows:

$$\begin{array}{rccc} F: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{S}\mathrm{et} \\ & C & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(C, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B), \end{array}$$

where $\mathrm{Hom}_{\mathcal{C}}(C, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B)$ is the standard Cartesian product of sets. For this functor to be representable, there must be an object Z in $\mathcal{C}$ such that, for every other object C in $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(C, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B) \overset{\text{in } \mathcal{S}\mathrm{et}}{\simeq} \mathrm{Hom}_{\mathcal{C}}(C, Z).$$

Rewording it, we need a bijection $(f_A, f_B) \longleftrightarrow h$. And this looks like what we wanted.

…and the projections? And the commutative diagrams?! Here they come: first of all, the reader, if they have arrived all the way to this point, should have already started to complain with something sounding like "Hey! What is F doing with the morphisms? We can't forget them!". Indeed: given a morphism $f : C \leftarrow D$ (sorry, it is controvariant), we associate to it the function

$$F(f): \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C,A) \times \mathrm{Hom}_{\mathcal{C}}(C,B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(D,A) \times \mathrm{Hom}_{\mathcal{C}}(D,B) \\ (f_A, f_B) & \longmapsto & (f_A \circ f, f_B \circ f) \end{array}. \tag{1}$$

Now we can complete the proof. First of all, we take as C the object Z itself. We obtain, buy construction,

$$\mathrm{Hom}_{\mathcal{C}}(Z,A) \times \mathrm{Hom}_{\mathcal{C}}(Z,B) \overset{\text{in } \mathcal{S}\mathrm{et}}{\simeq} \mathrm{Hom}_{\mathcal{C}}(Z,Z).$$

In the right-hand set, we certainly have a special element, that is the identity $\mathbb{1}_Z$. To it are then associated by F a pair of morphisms that we shall denote by (p_A, p_B), even if, at the moment, we do not know if they have the property required by Definition (6.18). Now we take again a generic object C and a pair of morphisms (f_A, f_B). The latter is an element of $\mathrm{Hom}_{\mathcal{C}}(Z,A) \times \mathrm{Hom}_{\mathcal{C}}(Z,B)$, hence a morphism $h : C \to Z$. This is the same as a morphism $h : Z \to C$ in $\mathcal{C}^{\mathrm{op}}$, so we can apply F to it and obtain a function as in Diagram (1). Putting everything together:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Z,A) \times \mathrm{Hom}_{\mathcal{C}}(Z,B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C,A) \times \mathrm{Hom}_{\mathcal{C}}(C,B) \\ (p_A, p_B) & \longmapsto & (p_A \circ h, p_B \circ h) \\ \updownarrow & & \| \\ & & (f_A, f_B) \\ & & \updownarrow \\ \mathbb{1}_Z & \longmapsto & h \\ \mathrm{Hom}_{\mathcal{C}}(Z,Z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C,Z) \end{array}.$$

q.e.d.

Remark 6.21 *The projections, in the definition of a product, are part of the definition. Changing them is equivalent, as shown above, to take a different isomorphism* $h_{\mathcal{C}}(Z) \cong F$.

Notation 6.22 *Thanks to the previous proposition, we can now denote a generic product of two objects A and B by $A \times B$ (or, very often, also $A \prod B$), the morphisms p_A and p_B by π_A and π_B and call them* projections.

Exercise 6.23 *Using different pairs of projections, make $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the product, in $\mathcal{A}b$, of two copies of $\mathbb{Z}/2\mathbb{Z}$ (if you are more intrepid, do the same with $\mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$).*

Example 6.24

1. *In $\mathcal{S}\mathrm{et}$, the categorical product is the Cartesian product, with the standard projections.*

2. *In $\mathcal{A}b$ (and in $\mathcal{G}\mathrm{r}$), the (categorical) product $A \times B$ is the direct sum $A \oplus B$.*

3. *In $\mathcal{T}\mathrm{op}$, given two topological spaces (X, τ_X) and (Y, τ_Y) is $(X \times Y, \tau_{X \times Y})$, where $X \times Y$ is, as a set, the Cartesian product, while $\tau_{X \times Y}$ is the coarsest topology that makes the projections continuous. By definition, this is called the* product topology. *When dealing with finitely many (two in this case) topological spaces, this is equal to the so called* box topology, *that is the topology generated by the basis of open subsets $\pi_X^{-1}(U) \times \pi_Y^{-1}(V)$ where $U \in \tau_X$ and $V \in \tau_Y$. When we take a product of infinitely many topological spaces though, the box topology is finer than the product topology.*

4. *In $\mathcal{R}\mathrm{ing}$, or $\mathcal{R}\mathrm{ng}$, the product or R and S is the cartesian product $R \times S$ with the operations defined component-wise.*

5. *In $(S, \leq)$, the category defined by a poset S, the product of two objects $s, t \in S$ is, if it exists, $\inf(s,t)$.*

Exercise 6.25 *Prove explicitly the statement in Example (6.24.(5)).*

A sort of associativity holds for products:

Proposition 6.26 *Let A, B, and C be three objects in a category $\mathcal{C}$, then we have a unique isomorphism*

$$A \times (B \times C) \simeq (A \times B) \times C.$$

that commutes with the projections.

Exercise 6.27 *Prove Proposition (6.26).*

The last proposition is a way to justify the notation $A \times B \times C$ and to introduce the following definition:

Definition 6.28 *A category $\mathcal{C}$ is said* to have finite products *if for every finite family of objects $\{A_i\}_{i \in I}$ in $\mathcal{C}$, indexed by a finite set I, there exists a product $\left(\prod_{i \in I} A_i, \{\pi_i\}_{i \in I}\right)$. Likewise, it is said* to have infinite products *if the same is true for any index set I.*

Remark 6.29 *Note that the product of an empty family of objects is a terminal object, if $\mathcal{C}$ has any, of course.*

Example 6.30

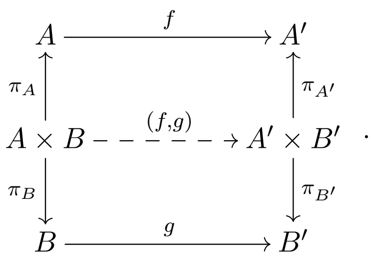
1. *The categories $\mathcal{S}\mathrm{et}$, $\mathcal{A}b$, Vect_k, $\mathcal{T}\mathrm{op}$ all have infinite products. The fact that $\mathcal{S}\mathrm{et}$ has infinite products, actually allows us to repeat the contruction, or definition, of products as representatives of functors in the category $\hat{\mathcal{C}}$.*

2. *The categories Vect_k^f and $\mathcal{A}b^f$ only have finite products.*

Products have many properties, we summarize some of them, in the form of an exercise, here:

Exercise 6.31 *Let $\mathcal{C}$ be a category and A, A', B, and B' objects of $\mathcal{C}$. Assume the products $A \times B$ and $A' \times B'$ exist.*

1. *Prove that if $f : A \to A'$ and $g : B \to B'$ are morphisms, then there exists a unique morphism $(f, g) : A \times B \to A' \times B'$ such that everything commutes in the following diagram:*



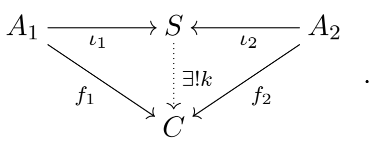
2. *With the notation of (1), prove that if f and g are monomorphisms, also (f, g) is a monomorphism.*

3. *Assume T is a terminal object in $\mathcal{C}$. Prove that $A \times T \simeq A$.*

After products, we want to define *coproducts*, that are also commonly called *sums*. We have an easy and lazy way out:

Definition 6.32 *Let $\{A_i\}_{i \in I}$ be a family of objects in a category $\mathcal{C}$. An object S, together with morphisms $\iota_i : A_i \to S$, is a* coproduct, or a sum, *of the A_i's in $\mathcal{C}$ if it is a product of the A_i's in $\mathcal{C}^{\mathrm{op}}$.*

As a result, we can transport every property of products to coproducts, reversing the arrows. For example, here is the defining commutative diagram, with only two objects:



Also, we can define the coproduct as a representative of the functor

$$\begin{array}{rccc} G: & \mathcal{C} & \longrightarrow & \mathcal{S}\mathrm{et} \\ & C & \longmapsto & \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(A_i, C) \end{array} .$$

We leave it to the reader to translate the propositions above to coproducts and we pass to examples:

Example 6.33

1. *In the category* $\mathcal{S}\mathrm{et}$, *the coproduct of two sets* X *and* Y, *is the disjoint union* $X \coprod Y$.

2. *In* $\mathcal{A}b$, *the coproduct is isomorphic to the product, so again we have the direct sum. It is worth to notice that, from the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & A \oplus B & \xleftarrow{\iota_B} & B \\ & {\scriptstyle f_A}\searrow & \downarrow{\scriptstyle k} & \swarrow{\scriptstyle f_B} & \\ & & C & & \end{array},$$

the homomorphism k *is computed, on a pair* (a, b) *as* $f_A(a) + f_B(b)$.

3. *In* $\mathcal{G}\mathrm{r}$, *the coproduct is suddenly much more complicated. A reason is simply that, since the operation is not commutative anymore, the homomorphism of the previous example, that in multiplicative notation would be* $f_G(g) \cdot f_H(h)$, *is defined, but different and no better than* $f_H(h) \cdot f_G(g)$. *Indeed, none of the two works. So given two groups* G *and* H, *with presentations respectively* $< S_G, R_G >$ *and* $< S_H, R_H >$, *we have to take the so called* free product, *that is the group with presentation* $< S_G \coprod S_H, R_G \coprod R_H >$. *Despite the name, this happens to be the coproduct in* $\mathcal{G}\mathrm{r}$.

4. *In the category* $\mathcal{R}\mathrm{ing}$, *the coproduct of two rings* A *and* B *is* $A \otimes_{\mathbb{Z}} B$. *More in general, if* A *and* B *are* $R-$*algebras, then we take* $A \otimes_R B$.

5. *In* $(S, \leq)$, *the category defined by a poset* S, *the coproduct of two objects* $s, t \in S$ *is, if it exists,* $\sup(s, t)$.

7 Adjoints – "The revenge of the forgetful functors"

"Adjoint functors arise everywhere."

S. Maclane, from the preface of [2]

As you might guess from the quote above, this is a very important topic in Category Theory. Before diving into it, we need to give two definitions:

Definition 7.1 *Given two categories* $\mathcal{C}$ *and* $\mathcal{C}'$, *the* product category $\mathcal{C} \times \mathcal{C}'$ *is the product in the category* $\mathcal{C}\mathrm{at}$ *and it is exactly what one would expect:*

$$\begin{aligned} \mathrm{Obj}(\mathcal{C} \times \mathcal{C}') &= \quad \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}'), \\ \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X, X'), (Y, Y')) &= \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}'}(X', Y'), \\ \circ^{\mathcal{C}\times\mathcal{C}'} &= \quad \textit{defined component-wise.} \end{aligned}$$

Definition 7.2 *Given categories* $\mathcal{C}$, $\mathcal{C}'$, *and* $\mathcal{D}$, *a functor* $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ *is traditionally called a* bifunctor.

Having a bifunctor is equivalent to have that $F(X, -) : \mathcal{C}' \to \mathcal{D}$ and $F(-, X') : \mathcal{C} \to \mathcal{D}$ are functors for every fixed object $X \in \mathcal{C}$ or $X' \in \mathcal{C}'$ and that, for every pair of morphisms $f : X \to Y$ and $f' : X' \to Y'$, the following diagram commutes:

$$\begin{array}{ccc} F(X, X') & \xrightarrow{F(X,f')} & F(X, Y') \\ {\scriptstyle F(f,X')}\downarrow & & \downarrow{\scriptstyle F(f,Y')} \\ F(Y, X') & \xrightarrow{F(Y,f')} & F(Y, Y') \end{array}.$$

Indeed, by definition, we have

$$(\mathbb{1}_Y, f') \circ (f, \mathbb{1}_{X'}) = (f, f') = (f, \mathbb{1}_{Y'}) \circ (\mathbb{1}_X, f'),$$

appying a bifunctor F to these identities, we get the above commutative diagram.

Example 7.3

1. *Given any category $\mathcal{C}$, we have a bifunctor* $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}\mathrm{et}$*, by Definition (1.1) and, adapting the new notation, always the same diagram as in Section 5:*

$$X \overset{f}{\leftarrow} Y \quad X' \overset{f'}{\rightarrow} Y' \quad \begin{array}{ccc} \alpha & \longmapsto & f' \circ \alpha \\ \Big\downarrow & \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, X') & \overset{f'^*}{\rightarrow} & \mathrm{Hom}_{\mathcal{C}}(X, Y') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Hom}_{\mathcal{C}}(Y, X') & \underset{f'^*}{\rightarrow} & \mathrm{Hom}_{\mathcal{C}}(Y, Y') \end{array} & \Big\downarrow \\ \alpha \circ f & \longmapsto & f' \circ \alpha \circ f \end{array}.$$

2. *Given a ring R, the tensor product is a bifunctor $\otimes_R : \mathrm{mod}{-}R \times R{-}\mathrm{mod} \to \mathcal{A}b$ (see also Example (5.7)).*

3. *If a category $\mathcal{C}$ has products, then $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, because of Exercise (6.31).*

Now we are ready:

Definition 7.4 *Given functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$, we say that L* is a left adjoint of R *(or viceversa) if there exists an isomorphism of bifunctors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}\mathrm{et}$*

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(L(-), -) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(-, R(-)),$$

that is a family $\{\varphi_{XY}\}_{X \in \mathcal{C}, Y \in \mathcal{D}}$ of isomorphisms in $\mathcal{S}\mathrm{et}$, that is bijections, such that

$$\varphi_{XY} : \mathrm{Hom}_{\mathcal{D}}(L(X), Y) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

functorial in X and Y.

There is really a lot to say about this definition, but that would go beyond the scope of a very short introduction. So here are the fundamentals:

Various remarks 7.5

1. *Since we can take any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, let's take in particular $Y = L(X)$. Then we get*

$$\varphi_{X, L(X)} : \mathrm{Hom}_{\mathcal{D}}(L(X), L(X)) \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(X, RL(X)).$$

Inside $\mathrm{Hom}_{\mathcal{D}}(L(X), L(X))$ *we do have a special element, that is $\mathbb{1}_{L(X)}$, so let's denote by η_X the morphism, in $\mathcal{C}$,*

$$\varphi_{X.L(X)}(\mathbb{1}_{L(X)}) : X \to RL(X).$$

Likewise, taking $X = R(Y)$, we get

$$\varphi^{-1}_{R(Y),Y} : \mathrm{Hom}_{\mathcal{D}}(LR(Y), Y) \overset{\sim}{\longleftarrow} \mathrm{Hom}_{\mathcal{C}}(R(Y), R(Y)).$$

and inside $\mathrm{Hom}_{\mathcal{C}}(R(Y), R(Y))$ *we take this time $\mathbb{1}_{R(Y)}$, and denote by ε_Y the morphism, in $\mathcal{D}$,*

$$\varphi^{-1}_{R(Y),Y}(\mathbb{1}_{R(Y)}) : LR(Y) \to Y.$$

But everything is functorial, thanks to the fact that φ is an isomorphism of bifunctors, hence we have actually obtained a morphism of functors

$$\eta = \{\eta_X : X \to RL(X)\}_{X \in \mathcal{C}} : \mathbb{1}_{\mathcal{C}} \to RL,$$

called unit of the adjunction, *and a morphism of functors*

$$\varepsilon = \{\varepsilon_Y : LR(Y) \to Y\}_{Y \in \mathcal{D}} : LR \to \mathbb{1}_{\mathcal{D}},$$

called counit of the adjunction.

2. ***Beware:*** *in some texts the naming is inverted! For example, we used the naming of [2] for unit and counit, but the notation of [1] for the functors, that is clearly better to remember them.*

3. *For every $X \in \mathcal{C}$, we have equalities*

$$\mathbb{1}_{L(X)} = L(X) \xrightarrow{L(\eta_X)} LRL(X) \xrightarrow{\varepsilon_{L(X)}} L(X),$$

that once again, since φ is functorial, gives the equality

$$\mathbb{1}_L = L \xrightarrow{L(\eta)} LRL \xrightarrow{\varepsilon_L} L$$

of natural transformations in $\mathrm{Hom}_{\mathcal{F}ct(\mathcal{C},\mathcal{D})}(L, L)$.

4. *Likewise, we have the equality*

$$\mathbb{1}_R = R \xrightarrow{\eta_R} RLR \xrightarrow{R(\varepsilon)} R$$

of natural transformations in $\mathrm{Hom}_{\mathcal{F}ct(\mathcal{D},\mathcal{C})}(R, R)$.

5. *The previous two remarks are important because they are equivalent to the existence of an adjunction:*

   Theorem 7.6 *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be two functors together with morphisms $\eta : \mathbb{1}_\mathcal{C} \to RL$ and $\varepsilon : LR \to \mathbb{1}_\mathcal{D}$ such that the equalities in (3) and (4) are true. Then L and R are adjoints.*

   Proof: *See for example [1, Prop.1.5.4 p.29] or [2, Theo.2 p.83], but do not forget the remark (2) above about notations.*

   q.e.d.

To shed some light on the definition, that is clearly not easy to digest, we give a lot of examples (and no proof), to prove (!?) that indeed "adjoint functors arise everywhere":

Example 7.7

0. *Every equivalence of category F gives an adjunction. Indeed, let G be a quasi-inverse of F and denote the necessary isomorphisms by $\varphi : G \circ F \to \mathbb{1}_\mathcal{C}$ and $\psi : F \circ G \to \mathbb{1}_\mathcal{D}$. Then we could use Theorem (7.6) using $\eta = \psi^{-1}$ and $\varepsilon = \varphi$. Alternatively, more directly:*

$$\mathrm{Hom}_\mathcal{D}(F(X), Y) = \mathrm{Hom}_\mathcal{D}(F(X), \mathbb{1}_\mathcal{D}(Y)) \simeq \mathrm{Hom}_\mathcal{D}(F(X), F \circ G(Y)) \simeq \mathrm{Hom}_\mathcal{C}(X, G(Y)),$$

where the last isomorphism holds because F is fully faithful (see Exercise (4.19)).

1. *Let's take back the forgetful functor $\rho : \mathcal{G}\mathrm{r} \to \mathcal{S}\mathrm{et}$ from Example (2.3.(1a)), and the functor $F : \mathcal{S}\mathrm{et} \to \mathcal{G}\mathrm{r}$, from Example (2.3.(2)), that constructs the free group over a set S. Then F is left adjoint to ρ: given a set S and a group G we have*

$$\mathrm{Hom}_{\mathcal{G}\mathrm{r}}(F(S), G) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}\mathrm{et}}(S, \rho(G)).$$

Indeed, defining a group homomorphism from the free group $F(S)$ to any other group G is the same as fixing the image of the generators, that is fixing any function from S to G, the latter being regarded as a set, so in fact $\rho(G)$.

Let's see what the unit and counit happen to be:

- (η) *The unit $\eta_S : S \to \rho(F(S))$ is just the inclusion of the generators (the elements of S) inside the free group over S (considered as a set).*
- (ε) *The counit $\varepsilon_G : F(G) \to G$ is the group homomorphism that projects $F(G)$ onto G, so its kernel is constituted by all the relations of G.*

2. *Exactly as before, we have that the functor F^{ab}, that associates to a set S the free abelian group over S, is left adjoint of the forgetful functor $\rho : \mathcal{A}b \to \mathcal{S}\mathrm{et}$.*

3. *We now consider Example (2.3.(1b)), with the forgetful functor $\rho : \mathcal{A}b \to \mathcal{G}\mathrm{r}$. This also has a left adjoint, that is the functor $(-)^{\mathrm{ab}}$ that associates to a group its abelianization, that is the quotient $G/[G, G]$. In other words, given a group G and an abelian group A, we have:*

$$\mathrm{Hom}_{\mathcal{A}b}(G/[G, G], A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}\mathrm{r}}(G, \rho(A)).$$

4. *Another forgetful functor with an interesting adjoint is* $\rho : R - \text{mod} \to \mathcal{A}b$*: given an* $R-$*module* M *and an abelian group* A*, we have*
$$\text{Hom}_R(R \otimes_{\mathbb{Z}} A, M) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(A, \rho(M)).$$
This is also called informally the "extension of scalars".

5. *And another one: take the forgetful functor* $\rho : \mathcal{F}\text{ld} \to \text{IntDom}$ *from the category of fields to the category of integral domains (with the obvious morphisms). Given a field* F *and an integral domain* D *we then have*
$$\text{Hom}_{\mathcal{F}\text{ld}}(Q(D), F) \xrightarrow{\sim} \text{Hom}_{\text{IntDom}}(D, \rho(F)),$$
where $Q(D)$ *is the field of fractions of* D*.*

6. *After having avenged the forgetful functors, we can go on. Take a category* $\mathcal{C}$ *and its* diagonal functor*:*
$$\begin{array}{rccc} \Delta : & \mathcal{C} & \to & \mathcal{C} \times \mathcal{C} \\ & X & \mapsto & (X, X) \end{array},$$
acting on the morphisms component-wise.

   If we now search for a right adjoint, it means that, given an X *in* $\mathcal{C}$ *and a pair* (Y, Z) *in* $\mathcal{C} \times \mathcal{C}$*, we want a bifunctor* $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *filling up the isomorphism*
$$\text{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(X), (Y, Z)) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X, ?(Y, Z)).$$
Looking at it, the elements of $\text{Hom}_{\mathcal{C}\times\mathcal{C}}(\Delta(X), (Y, Z))$ *are pairs* (f, g) *of morphisms in* $\mathcal{C}$*, more specifically* $f : X \to Y$ *and* $g : X \to Z$*. To such a pair, we need to associate a morphism* $X \to ?(Y, Z)$*. We hope someone guessed that* $Y \times Z$ *is the answer and* $- \times -$ *(if it exists in* $\mathcal{C}$*) is the bifunctor we are searching for. But where have the projections* π_Y *and* π_Z *gone?! Let's not forget the unit and counit:*

   (η) *The unit* $\eta_X : X \to \Delta(X) = X \times X$ *is just the* diagonal morphism*;*

   (ε) *The counit is more interesting:* $\varepsilon_{(Y,Z)} : \Delta(Y \times Z) = (Y \times Z) \times (Y \times Z) \to (Y, Z)$ *is a pair morphisms, one is* $(Y \times Z) \to Y$ *and the other is* $(Y \times Z) \to Z$*. Here are the projections* π_Y *and* π_Z*, because the existence of the isomorphism of bifuctors* φ *ensure they behave as we want.*

   What if we search for a left adjoint of Δ*? Unsurprisingly, we have that its left adjoint is the bifunctor* $\coprod$*, if products exist in* $\mathcal{C}$*.*

7. *Let's take again a category* $\mathcal{C}$ *and define the category* $\mathbf{1}$ *with one object* $\{*\}$ *and the sole identity as morphism. Define the functor 1 on the objects as*
$$\begin{array}{rccc} 1 : & \mathcal{C} & \to & \mathbf{1} \\ & X & \mapsto & \{*\} \end{array}$$
and sending every morphisms f *in* $\mathcal{C}$ *to the identity* $\mathbb{1}_{\{*\}}$*. This functor has a right adjoint* R *if an only if the category* $\mathcal{C}$ *has a terminal object. In this case,* $R(\{*\})$ *is indeed a terminal object. This should be clear by the isomorphism that need to be satisfied:*
$$\{\mathbb{1}_{\{*\}}\} = \text{Hom}_{\mathbf{1}}(\{*\}, \{*\}) = \text{Hom}_{\mathbf{1}}(1(X), \{*\}) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X, R(\{*\})).$$
Likewise, the same functor $\mathbf{1}$ *has a left adjoint* L *if an only if the category* $\mathcal{C}$ *has an initial object. In this case,* $L(\{*\})$ *is indeed an initial object.*

This long list of examples should be enough to convince you that the concept of adjoint is actually everywhere in Mathematics. We conclude these notes, at least for the moment, with few important propositions about adjoints.

Proposition 7.8 *If* $L : \mathcal{C} \to \mathcal{D}$ *is left adjoint of* $R : \mathcal{D} \to \mathcal{C}$ *and* $L' : \mathcal{D} \to \mathcal{E}$ *is left adjoint of* $R' : \mathcal{E} \to \mathcal{D}$*, then* $L'L$ *is left adjoint of* RR'*.*

Proof: This is just a composition of isomorphisms of bifunctors.

q.e.d.

Proposition 7.9 *If* L *has two right adjoints* R *and* R'*, then* R *and* R' *are isomorphic via a unique isomorphism. The same is true for left adjoints.*

Exercise 7.10 *Prove the proposition above.*

Corollary 7.10.1 $(-)^{\text{ab}} \circ F = F^{\text{ab}}$.

References

[1] M. Kashiwara, P. Shapira *Categories and Sheaves*, Springer 2006.

[2] S. Mac Lane *Categories for the Working Mathematician* Springer Second edition 1971

[3] A. Hatcher *Algebraic Topology* Cambridge University Press (available online) 2002