PROJECTIVE GEOMETRY

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Lecture 1. Projective spaces

Intuitive definition. Consider a 3-dimensional vector space V and some plane π which does not pass through the origin (i.e. the zero vector). Lines through the origin (i.e. 1-dimensional subspaces of V) which are not parallel to the plane π will intersect π in a point, and planes through the origin (i.e. 2-dimensional subspaces of V) which are not parallel to π will meet π in a line. The plane π consisting of these points and these lines can serve as your first idea of what a projective space (in this case a projective plane) is.

Question 1. *Do you see that each two lines of π have at least a point of π in common?*

Analytic definition. This definition does not cover all of the projective spaces, as there are many projective planes which do not satisfy this definition. However, it is the correct definition for projective spaces of higher dimension (although this is far from trivial), and it serves well for most commonly used projective planes (the ones which can be *coordinatised* by (skew-)fields, later more about this).
Consider a vector space V. Define the projective space $\mathbb{P}V$ as the set of subspaces of V of dimension and co-dimension at least one. Subspaces of dimension one of V are called *points* of $\mathbb{P}V$. If V has dimension two then $\mathbb{P}V$ is called a *projective line* and it just consists of points. If V has dimension larger than one, then the subspaces of dimension two of V are called *lines* of $\mathbb{P}V$.

Question 2. *Can you now prove that each two lines of a projective plane meet in a point?*

Notation. Not all mathematicians use the same notation for a projective space. If V is of dimension $n+1$ over some field $\mathbb{F}$ then instead of $\mathbb{P}V$ we also write $\mathbb{P}^n$ (or $\mathbb{P}^n_{\mathbb{F}}$ if we want to emphasize which field we are working over). If the field is finite, i.e. $\mathbb{F} = \mathbb{F}_q$ for some prime power q, then it is common practice to use $\mathrm{PG}(n,q)$, but we will stick to $\mathbb{P}^n$ or use the alternative notation $\mathbb{P}^n_q$. Also $\mathrm{PG}(V)$ is frequently used instead of $\mathbb{P}V$, especially in finite geometry.

Projective lines. A projective line is just a set of points, so there is no "geometry" to it. So, although projective lines are important and show up in every other projective space, we will have to exclude it from the axiomatic definition, which only works for projective spaces of dimension at least two.

Synthetic definition. Forget (for a moment) everything we just talked about. Consider two sets $\mathcal{P}$ and $\mathcal{L}$; call their elements *points* and *lines*. In addition to these two sets, consider a symmetric relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L} \cup \mathcal{L} \times \mathcal{P}$ (which we will call the *incidence* relation). A point p and a line ℓ are called *incident* if $(p,\ell) \in \mathcal{I}$, and we also say that p lies on ℓ, ℓ contains p, etc.

Such a triple $(\mathcal{P},\mathcal{L},\mathcal{I})$ is called a *projective plane* if the following three axioms are satisfied: (pp1) every two points span a unique line; (pp2) every two lines meet in a unique point; (pp3) there exist 4 points, no three of them on a line. (Use your intuition to understand what is meant by *span* and *meet*.)

Question 3. *Can you prove that $\mathbb{P}^2$ satisfies these three axioms?*

For projective spaces of dimension ≥ 2 we need a slightly different second axiom. A *projective space* $\mathbb{P}$ is a *thick* (at least three points on each line) point-line geometry $(\mathcal{P},\mathcal{L},\mathcal{I})$ together with its subspaces, satisfying the following axioms: (ps1) every two points span a unique line; (ps2) (Veblen) every line intersecting two sides of a triangle, not through the vertices, intersects also the third side; (ps3) there exists a triangle.

Subspaces and dimension. Continue with the projective space $\pi = (\mathcal{P},\mathcal{L},\mathcal{I})$ as defined above. A subset U of π is called a *subspace* of π if for each two distinct points $x, y \in U$ the line spanned by x and y has all its points in U. It follows that each point and each line of π is a subspace of π. The incidence relation $\mathcal{I}$ is extended to a relation between all subspaces of π, where two subspace are *incident* if one of them contains the other. The *dimension* of a projective space π (or subspace) is the length of the longest *chain of nested subspaces*, where as a convention, we define the *empty subspace* as a subspace of π incident with every subspace. So a point has dimension zero; a line dimension one; a plane dimension 2; a solid dimension 3; etc.

Question 4. *Can you show that each subspace (which is not a point or a line) is itself a projective space?*

Exercise 1. *Prove that for projective planes, the set of (pp)-axioms and the set of (ps)-axioms are equivalent.*

Projective planes. Two famous examples of projective planes are the Fano plane (of *order 2*) and the Moulton plane.

Question 5. *Can you draw a picture of the projective plane* $\mathbb{P}V$ *where* $V = \mathbb{F}_3^3$*?*

Affine spaces. An affine space is obtained from a projective space by removing a hyperplane. An affine space can also be defined by an analytic definition, as the space consisting of all translates of all subspaces of a given vector space V, where the translates of a subspace $U \leq V$ are the sets $v + U = \{v + u : u \in U\}$, $v \in V$. We denote the affine space obtained from a projective space π by removing a hyperplane H of π by $\mathbb{A}(\pi, H)$, and the affine space obtained from a vector space V by $\mathbb{A}V$ (or $\mathbb{A}^n$ etc.).

Projective completion. Given an affine space $\mathbb{A}^n$, one can define a projective space as the space consisting of all subspaces of $\mathbb{A}^n$ (the *affine subspaces*) together with a *new subspace* for each parallel class of subspaces of $\mathbb{A}^n$. Natural incidence makes this into a projective space, called the *projective completion of* $\mathbb{A}^n$, which we denote by $\overline{\mathbb{A}^n}$.

It is not so difficult to see that the projective completion is unique. However, the answer to the question whether any two affine spaces obtained from a given projective space are *isomorphic* is more complicated. For projective spaces $\mathbb{P}V$ the answer is affirmative, but for non-classical projective planes the answer depends on the properties of the plane.

Synthetic definition. Similarly, as for a projective planes, an affine plane can be defined by the properties (ap1) every two lines span a unique line; (ap2) for every anti-flag (p, L) there exists a unique line through p not meeting L; (ap3) there exists a triangle.

Erlangen. The Erlangen program proposes projective geometry as the main type of geometry of which other well-known geometries are special cases. As we will explain in this lecture, affine geometry, for example, is less general and can be seen as part of projective geometry. The name of this program refers to the Erlangen University in Germany where the mathematician Felix Klein proposed this unification of the different kind of geometries in 1872.

Exercise 2. *Show that every line of a projective plane has the same cardinality.*

This cardinality is equal to $1 + |\mathbb{F}|$, i.e. one plus the size of the field $\mathbb{F}$, in the case that the projective plane is $\mathbb{P}V$ for $V = \mathbb{F}^3$. In the case of a finite projective plane this cardinality minus one is called the order of the projective plane. So the Fano plane is a projective plane of order 2.

Exercise 3. *Show that there is a unique projective plane of order 2, and a unique projective plane of order 3.*

Lecture 2. Collineations

Morphisms. We have defined projective spaces, which will be our object of study. The next thing mathematicians do is to try and understand the morphisms between these objects. This is crucial if one ever wants to obtain sensible classification results. Compare this to group theory. A group whose elements are matrices and another group whose elements are permutations might be "isomorphic", i.e. they might be different ways of representing the same "abstract group". The morphisms between projective spaces that we are interested in are called *collineations*.

Collineations. A *collineation* between two projective spaces π and π' of dimension ≥ 2 is a bijection between the set of subspaces of $\mathcal{P}$ and the set of subspaces of $\mathcal{P}'$ which preserves dimension and incidence.

Projectivities. Any element $A \in \mathrm{GL}(V)$ defines a collineation α of $\mathbb{P}V$, where $V = \mathbb{F}^{n+1}$. Such a collineation is called a *projectivity.* The action of α on $\mathbb{P}V$ is completely determined by its action on the points of $\mathbb{P}V$, which we define as follows. The image of a point $\langle x \rangle$ under α is the point $\langle y \rangle$ where $y^T = Ax^T$. The *projectivity group* of $\mathbb{P}V$ is denoted by $\mathrm{PGL}(V)$ or $\mathrm{PGL}_{n+1}(\mathbb{F})$. Fixed points of α correspond to eigenvectors of A.

Question 6. *Can you find a collineation which is not a projectivity?*

Frames. An *arc* in a projective space of dimension n is a set of points no $n+1$ of which are contained in a hyperplane (i.e. the points are in *general position*). A *frame* is an ordered arc of size $n+2$.

Theorem 7. *The group* $\mathrm{PGL}(V)$ *acts sharply transitively on the set of frames of* $\mathbb{P}V$*.*

Proof. Consider two frames Γ and Γ' in $\mathbb{P}V$ and choose bases B and B' for V, each corresponding to the first $\dim V$ points of Γ and Γ'. The matrix $A \in \mathrm{GL}(V)$ mapping B to B' induces a projectivity mapping the first $n+1$ points of Γ to the first $n+1$ points of Γ'. So w.l.o.g. we may assume the first $n+1$ points of both Γ and Γ' correspond to the standard basis of V. Any projectivity fixing these points is induced by a diagonal matrix with nonzero elements on the diagonal. Also, for any point p of $\mathbb{P}V$ which does not have any zero coordinate, there exists a unique projectivity α of that form (with the coordinates of p on the diagonal) which maps the point p_{n+1} with coordinates $(1, 1, \ldots, 1)$ (the all one vector) to p. Since the $(n+2)$-nd point of Γ (and of Γ') cannot have any zero coordinates, the action of $\mathrm{PGL}(V)$ is sharply transitive. $\square$

Canonical forms. We know from linear algebra that a change of basis in V changes the matrix of the linear transformation A into $C^{-1}AC$ for some $C \in \mathrm{GL}(V)$. This gives a limited number of canonical forms for projecitivities of $\mathbb{P}V$. If we work over the complex numbers $\mathbb{F} = \mathbb{C}$ then we obtain three *Jordan canonical forms* for projectivities in $\mathbb{P}^1$:

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \; A_2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \text{ and } A_3 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

If α_i denotes the projectivity induced by A_i $(i = 1, 2, 3)$ then α_1 is the identity in $\mathrm{PGL}_2(\mathbb{C})$, α_2 has two fixed points, and α_3 has one fixed point. The same exercise for $\mathbb{P}^2$ gives six different types of projectivities in $\mathrm{PGL}_3(\mathbb{C})$.

Dual action. The projectivity α induced by $A \in \mathrm{GL}(V)$ maps the hyperplane with equation $ax^T = 0$ to the hyperplane $bx^T = 0$ where $b = aA^{-1}$.

Principle of duality. Analogous to the notion of a dual vector space we have a dual projective space $\mathbb{P}^\vee$. The points of the dual space $\mathbb{P}^\vee$ are the hyperplanes of $\mathbb{P}$, etc.

The dual of the simple statement that "two points in a projective plane are on a unique common line" is "two lines in a projective plane intersect in a unique point", which implies that there are no parallel lines in a projective plane. This "dualising" turns out to be very useful in general, and it became know as the "*principle of duality*":

Each statement, configuration or theorem in a projective space (in terms of points, lines, ...) has a dual *statement or configuration (in terms of hyperplanes, subspaces of co-dimension two, ...)*

For example, recalling that an arc in $\mathbb{P}^2$ is a set of points in a projective plane $\mathbb{P}^2$, no three of which are collinear, we obtain the notion of a *dual arc* in $\mathbb{P}^2$: a set of lines, no three of which are concurrent.

Semilinear transformations. If the field $\mathbb{F}$ allows non-trivial automorphisms, then the existence of collineations which are not projectivities follows by considering the induced action of a *semilinear transformation* $v^T \mapsto A(v^T)^\sigma$ on the projective space. The semilinear group is denoted by $\Gamma\mathrm{L}(V)$ and the projective semilinear group by $\mathrm{P}\Gamma\mathrm{L}(V)$. The following theorem is one of the fundamental theorems of projective geometry.

Theorem 8. *The collineation group of* $\mathbb{P}V$ *with dim*$V \geq 3$ *is* $\mathrm{P}\Gamma\mathrm{L}(V)$.

Proof. (i) First assume that $\alpha \in \mathrm{Aut}(\mathbb{P}V)$ fixes the standard frame $(p_0, \ldots, p_{n+1})$ pointwise.
(1) The point r_0 with coordinates $(0, 1, \ldots, 1)$ is the intersection of the line $\langle p_0, p_{n+1}\rangle$ with the hyperplane $\langle p_1, \ldots, p_n\rangle$, and hence $r_0^\alpha = r_0$. The point r_1 with coordinates $(0, 0, 1, \ldots, 1)$ is the intersection of the line $\langle p_1, r_0\rangle$ with the subspace $\langle p_2, \ldots, p_n\rangle$, and hence $r_1^\alpha = r_1$. Continuing in this way, one obtains that each point with coordinates belonging to $\{0, 1\}$ is fixed by α.

(2) Now consider the point p with coordinates

$$(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = (\overbrace{0, 0, \ldots, 0}^{a}, \overbrace{1, 1, \ldots, 1}^{b}, \overbrace{-1, -1, \ldots, -1}^{c}).$$

The point p is the intersection of the line

$$\ell := \langle(\overbrace{1, 1, \ldots, 1}^{a+b}, \overbrace{0, 0, \ldots, 0}^{c}), (\overbrace{1, 1, \ldots, 1}^{a}, \overbrace{0, 0, \ldots, 0}^{b}, \overbrace{1, 1, \ldots, 1}^{c})\rangle$$

with the subspace $U := \langle p_a, \ldots, p_n\rangle$. It follows that also p is fixed by α. Similar arguments can be used to conclude that each point with coordinates belonging to $\{0, 1, -1\}$ is fixed by α.

(3) Next consider all points on the line p_0p_1 but different from p_1, i.e. all points with coordinates belonging to the set $\{(1, z, 0, \ldots, 0) \ : \ z \in \mathbb{F}\}$. Since α fixes the line $\langle p_0, p_1\rangle$ and the points with coordinates from $\{0, 1, -1\}$ it follows that the image under α of a point $(1, z, 0, \ldots, 0)$ is $(1, z^\theta, 0, \ldots, 0)$ for some bijection θ of $\mathbb{F}$ with $0^\theta = 0$, $1^\theta = 1$, and $-1^\theta = -1$. The point with coordinates $(1, 0, z, 0, \ldots, 0)$ is the intersection of the line

$$\langle(1, z, 0, \ldots, 0), (0, -1, 1, 0 \ldots, 0)\rangle$$

with the line $\langle p_0, p_2\rangle$, and is therefore mapped onto the point $(1, 0, z^\theta, 0, \ldots, 0)$ by α. Similarly, one obtains that each point with coordinates $e_0 + z_ie_i$ is mapped onto the point with coordinates $e_0 + z^\theta e_i$ by α.

(4) Now consider the hyperplanes Π_i with equations $z_iX_0 - X_i = 0$. Each such hyperplane Π_i is spanned by the point with coordinates $e_0 + z_ie_i$ together with all but one of the points $p_1, \ldots, p_n$, and hence is mapped onto the hyperplane with equation $z_i^\theta X_0 - X_i = 0$. The point with coordinates $(1, z_1, \ldots, z_n)$ is the intersection of all these hyperplanes and is therefore mapped onto the point with coordinates $(1, z_1^\theta, \ldots, z_n^\theta)$. Using the same arguments one shows that each point with normalised coordinates $(0, \ldots, 0, 1, z_i, \ldots, z_n)$ satisfies

$$\langle(0, \ldots, 0, 1, z_i, \ldots, z_n)\rangle^\alpha = \langle(0, \ldots, 0, 1, z_i^\theta, \ldots, z_n^\theta)\rangle.$$

(5) Now we show that θ is an automorphism of $\mathbb{F}$. Recall that we already proved that $0^\theta = 0$, $1^\theta = 1$, and $-1^\theta = -1$. Consider the fact that the line

$$\langle(1, x, y, 0, \ldots, 0), (0, -1, 1, 0, \ldots, 0)\rangle$$

meets p_0p_2 in the point with coordinates $(1, 0, y + x, 0, \ldots, 0)$ and the line

$$\langle(1, y, 0, 0, \ldots, 0), (0, -1, x, 0, \ldots, 0)\rangle$$

meets p_0p_2 in the point with coordinates $(1,0,yx,0,\ldots,0)$. The above paragraphs imply that $(x+y)^\theta = x^\theta + y^\theta$ and $(xy)^\theta = x^\theta y^\theta$, for all $x, y \in \mathbb{F}$, i.e. $\theta \in \mathrm{Aut}(\mathbb{F})$.

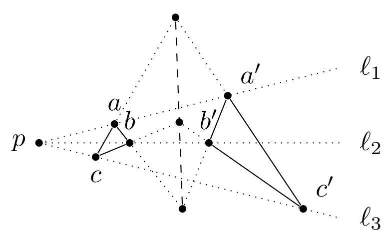
We conclude that $\alpha \in \mathrm{P\Gamma L}(n+1,\mathbb{F})$, in particular α is the collineation induced by the seminlinear map (I_{n+1},θ).

(ii) Now consider any collineation β of $\mathbb{P}V$. Then there exists a $\phi \in \mathrm{P\Gamma L}(n+1,\mathbb{F})$ which maps the frame $(p_0^\beta,\ldots,p_{n+1}^\beta)$ onto the standard frame $(p_0,\ldots,p_{n+1})$. Therefore $\beta\phi$ fixes the standard frame pointwise. Applying part (i) we obtain $\beta\phi \in \mathrm{P\Gamma L}(n+1,\mathbb{F})$, and hence $\beta \in \mathrm{P\Gamma L}(n+1,\mathbb{F})$. $\square$

Lecture 3. Desargues and Pappus

The following two theorems are classical results and are of particular historical interest in the study of projective planes. Their proofs serve as exercises to get familiar with *analytic projective geometry*. The study of projective planes became very popular in the 20th century, with the discovery of many so-called *non-desarguesian* projective planes. These are projective planes in which Desargues' Theorem does not hold.

Theorem 9 (Desargues)**.** *Consider three concurrent lines ℓ_1, ℓ_2 and ℓ_3 in $\mathbb{P}^n = \mathbb{P}(V)$, $n \geq 2$, and two triangles Δabc and $\Delta a'b'c'$ whose vertices a, b, c and a', b', c' lie on the lines ℓ_1, ℓ_2 and ℓ_3, respectively. Then the points $aa' \cap bb'$, $aa' \cap cc'$ and $bb' \cap cc'$ are collinear.*



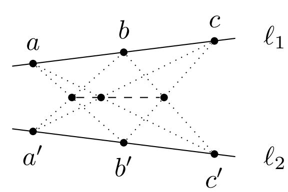
Proof. If the lines ℓ_1, ℓ_2 and ℓ_3 are not coplanar, then the theorem follows from the fact that the points $aa' \cap bb'$, $aa' \cap cc'$ and $bb' \cap cc'$ are contained in both of the planes $\langle a, b, c \rangle$ and $\langle a', b', c' \rangle$. These two planes are contained in the three-dimensional space $\langle \ell_1, \ell_2, \ell_3 \rangle$ and hence intersect in a line.

If the lines ℓ_1, ℓ_2 and ℓ_3 are coplanar, then we may restrict ourselves to that plane, i.e. assume that $n = 2$. Then the ordered 4-tuple (a, b, c, p) forms a frame, and we may assume that $a(1, 0, 0)$, $b(0, 1, 0)$, $c(0, 0, 1)$ and $p(1, 1, 1)$. Since a' lies on $\ell_1 = pa$, b' on $\ell_2 = pb$, and c' on $\ell_3 = pc$, there exist $\alpha, \beta, \gamma \in \mathbb{F} \neq \{0\}$ such that $a'(1 + \alpha, 1, 1)$, $b'(1, 1 + \beta, 1)$ and $c'(1, 1, 1 + \gamma)$. Calculating the intersection $aa' \cap bb'$ gives the point with coordinates $(\alpha, -\beta, 0)$. Similarly we obtain the points with coordinates $(0, \beta, -\gamma)$ and $(\alpha, 0, -\gamma)$. Since

$$\det \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & \beta & -\gamma \\ \alpha & 0 & -\gamma \end{bmatrix} = 0,$$

these three points are collinear. □

Theorem 10 (Pappus)**.** *Consider two lines ℓ_1 and ℓ_2 in $\mathbb{P}^n = \mathbb{P}(V)$, $n \geq 2$, and two triples (a, b, c) and (a', b', c') of points on the lines ℓ_1 and ℓ_2, respectively, but different from the intersection $\ell_1 \cap \ell_2$. Then the points $ac' \cap ca'$, $ab' \cap ba'$ and $bc' \cap cb'$ are collinear.*



Proof. We may assume w.l.o.g. that the points of the frame (a, b, a', b') have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, respectively. Then the points c and c' must have coordinates $(1, \alpha, 0)$ and $(\beta, \beta, 1)$ for some $\alpha, \beta \in \mathbb{F} \setminus \{0\}$. Calculating the points of intersection we obtain $(0, 1, 1)$, $(\beta, \alpha\beta, \alpha)$ and $(\beta, 1 - \alpha + \alpha\beta, 1)$. Since

$$\det \begin{bmatrix} 0 & 1 & 1 \\ \beta & \alpha\beta & \alpha \\ \beta & 1 - \alpha + \alpha\beta & 1 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 1 \\ \beta & \alpha\beta & \alpha \\ 0 & 1 - \alpha & 1 - \alpha \end{bmatrix} = 0$$

these points are collinear. □

It can be shown that Pappus implies Desargues, see [Seidenberg 1976] for a proof and historical comments. The configurations of Desargues and Pappus are of fundamental importance in the study of projective planes. We mention without proof that a projective plane in which Pappus' theorem holds true is isomorphic to some $\mathbb{P}^2_{\mathbb{F}}$, $\mathbb{F}$ a field, and a projective plane in which Desargues' theorem holds true is isomorphic to some $\mathbb{P}^2_K$ where K is a *skewfield* (a possibly non-commutative field).

André/Bruck-Bose. There is a beautiful geometric construction for projective planes. The construction starts with a partition S of $\mathbb{P}^{2n-1}$ by subspaces of dimension $n-1$, which is called an $(n-1)$-*spread* (or *spread* for short). Embed $\mathbb{P}^{2n-1}$ as a hyperplane in $\mathbb{P}^{2n}$ and consider the following incidence structure $\pi(S) = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. The "points" $\mathcal{P}$ of $\pi(S)$ are the points in $\mathbb{P}^{2n} \setminus \mathbb{P}^{2n-1}$, and the "lines" $\mathcal{L}$ of $\pi(S)$ are the n-dimensional subspaces of $\mathbb{P}^{2n}$ intersecting $\mathbb{P}^{2n-1}$ in an element of S.

Theorem 11. *The incidence structure $\pi(S)$ is an affine plane.*

Proof. One easily verifies the three axioms (ap1), (ap2), and (ap3). □

Spreads. Here is a simple construction of an $(n-1)$-*spread*. Consider an n-dimensional field extension K of a field $\mathbb{F}$. The projective line $\mathbb{P}^1_K$ becomes a $\mathbb{P}^{2n-1}_{\mathbb{F}}$. The points of $\mathbb{P}^1_K$ are 1-dimensional vector subspaces over K and therefore n-dimensional over $\mathbb{F}$. Projectively this gives us the desired $(n-1)$-spread of $\mathbb{P}^{2n-1}_{\mathbb{F}}$.

Derived planes. Non-desarguesian projective planes can be obtained as the so-called *derived planes* from $\mathbb{P}V$ with $V = \mathbb{F}^3$. It uses the *André-Bruck-Bose* constuction, and is obtained by replacing a regulus in the spread associated with $\mathbb{P}V$ by its opposite regulus.

Translation planes. The André/Bruck-Bose construction gives a particularly nice type of projective plane known as a *translation plane*: a projective plane π which contains a line L such that for each point p on L and each line $M \neq L$ through p the group of automorphisms of π fixing p and M acts transitively on the set of points on M distinct from p.

Lecture 4. The projective line

Although one could argue that the projective line is not all that interesting from a geometric point of view, since it merely consists of points, it is a subspace of every other projective space, and as such, it plays a fundamental role.

Collineations. (Recall that $\mathbb{P}^1$ was excluded in our previous definition of a collineation.) Collineations of $\mathbb{P}^1$ are defined as elements of $\mathrm{P\Gamma L}(V)$. Labelling the points of $\mathbb{P}^1$ by the elements of $\mathbb{F} \cup \{\infty\}$, a collineation of $\mathbb{P}^1$ gives a permutation of $\mathbb{F} \cup \{\infty\}$. The point $(1, 0)$ corresponds to ∞ and the point $(y, 1)$ corresponds to $y \in \mathbb{F}$. The collineation induced by the pair (A, θ) with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the map

$$y \mapsto \frac{ay^\theta + b}{cy^\theta + d}. \tag{1}$$

Cross ratio. The cross ratio is a "number" defined from a quadruple (p_1, p_2, p_3, p_4) of collinear points. If the first three points are distinct, then the cross ratio is equal to the *affine coordinate* of p_4 w.r.t. the frame (p_1, p_2, p_3). In order to find a formula for the cross ratio we will make use of Theorem 7. Suppose p_i has coordinates (x_i, y_i), and define d_{ij} as the determinant of the matrix whose rows are (x_i, y_i) and (x_j, y_j). The projectivity mapping the standard frame to (p_1, p_2, p_3) has matrix $A = \begin{bmatrix} \lambda x_1 & \mu x_2 \\ \lambda y_1 & \mu y_2 \end{bmatrix}$ where $\lambda = d_{32}/d_{12}$ and $\mu = d_{13}/d_{12}$, which has determinant $\lambda\mu d_{12}$. So in order to obtain the affine coordinate of p_4 with respect (p_1, p_2, p_3) we need to apply the inverse of A to the coordinates of p_4 and then divide the first by the second coordinate. This gives the following formula for the cross ratio of the quadruple (p_1, p_2, p_3, p_4)

$$\frac{d_{31}d_{42}}{d_{32}d_{41}}. \tag{2}$$

In order to make this definition somewhat more general, we let it apply to quadruples with two of its first three points equal, obtaining cross ratios 1 (if the first two points are the same), 0 (if the first point equals the third), and ∞ (if the last two points are equal). The following theorem comes for free.

Theorem 12. *Two (ordered) quadruples are projectively equivalent iff they have the same cross ratio.*

In other words, the cross ratio provides a *complete system of projective invariants* for collinear quadruples.

Theorem 13. *If (p_1, p_2, p_3, p_4) has cross ratio λ and $\alpha = \overline{\varphi}$ with $\varphi = (A, \theta)$, then the quadruple $(p_1^\alpha, p_2^\alpha, p_3^\alpha, p_4^\alpha)$ has cross ration λ^θ.*

Proof. Suppose that the cross ratio of the quadruple (p_1, p_2, p_3, p_4) is y.

First assume (p_1, p_2, p_3) is a frame. Let $\beta \in \mathrm{PGL}(V)$ be such that it maps the frame $(p_1^\alpha, p_2^\alpha, p_3^\alpha)$ to the (p_1, p_2, p_3). Then $\alpha\beta$ fixes the frame (p_1, p_2, p_3) and is therefore of the form $\overline{\varphi}$ where $\varphi = (I_2, \theta)$. Applying $\alpha\beta$ to the point $(y, 1)$ gives $(y^\theta, 1)$. So the cross ratio of the quadruple $(p_1^{\alpha\beta}, p_2^{\alpha\beta}, p_3^{\alpha\beta}, p_4^{\alpha\beta})$ is y^θ. This concludes the proof since β leaves the cross ratio invariant.

If (p_1, p_2, p_3) is not a frame, then the cross ration is either 0, 1 or ∞ in which case there is nothing to prove. □

Action of S_4. Permutating the points of a quadruple (by an element of S_4) gives an action of S_4 on the set of cross ratios which is obtained from a set of 4 points in $\mathbb{P}^1$.

Theorem 14. *The 24 quadruples obtained from the quadruple (p_1, p_2, p_3, p_4) by the action of the symmetric group S_4, produces six different values for the cross ratio*

$$\left\{\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{1-\lambda}\right\}.$$

Proof. It is clear from the formula (2) that interchanging the first (or last) two points gives $1/\lambda$. This implies that the permutations (12) and (34) map the cross ratio to its inverse. It also implies that the orbit of λ under S_4 has size at most six. Changing the second and third point can be obtained by the element of $\mathrm{PGL}(V)$ induced by the matrix $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ which maps the point $(y, 1)$ to $(1-y, 1)$. This shows that the permutation $(23) \in S_4$ maps the cross ratio λ to $1-\lambda$. Since these permutations generate S_4 we obtain the 6 values which are listed. $\square$

A quadruple having cross ratio equal to -1 is called *harmonic*. They are characterised by the fact that their cross ratio is invariant under the permutation $(12) \in S_4$.

A complete invariant. For any quadruple T of distinct points in $\mathbb{P}^1$, with cross ratios $\lambda_1, \dots, \lambda_6$, define

$$J(T) = \sum_{i<j} \lambda_i \lambda_j.$$

Theorem 15. *For each $\lambda \in \{\lambda_1, \dots, \lambda_6\}$*

$$J(T) = 6 - \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

Proof. By definition, $J(T)$ is the coefficient of X^4 in the polynomial $\prod_{i=1}^{6}(X - \lambda_i)$, which means that if the expression is valid for one of the λ_i's, then it holds for all λ_i's. (One can also show that the equation holds true for the other values of λ by observing that both of the substitutions $\lambda \mapsto 1/\lambda$ and $\lambda \mapsto 1-\lambda$ leaves $J(T)$ invariant.) To verify the expression, put $\lambda = \lambda_1$ and write the other λ_i's in terms of λ. Compute $J(T)$. $\square$

Let Ω denote the set of quadruples consisting of distinct points for which the six values of the cross ratios obtained from them by the action of S_4 are all distinct.

Theorem 16. *Two quadruples $T, T' \in \Omega$ are* $\mathrm{PGL}(V)$*-equivalent if and only if $J(T) = J(T')$.*

Proof. If two quadruples T and T' are equivalent then there must be some ordering of their points for which the cross ratios are equal. This means that the two sets of six cross ratios obtained from both quadruples by permuting the position of the points must be the same. This implies $J(T) = J(T')$.

Assume that the six cross ratios obtained from T are distinct. It follows from the above that these are the six distinct roots of the degree six polynomial $X^2(X^2-1)(6 - J(T)) + (X^2 - X + 1)^3$. Therefore, the six values of the cross ratios obtained from the quadruple T are uniquely determined by the value of $J(T)$. In other words, if $J(T) = J(T')$, then T and T' determine the same set of six distinct values for the cross ratio, and are thus equivalent under $\mathrm{PGL}(V)$. The case in which not all six cross ratios obtained form T are distinct is left as an exercise. $\square$

If the six values of the cross ratios obtained from a quadruple T consisting of distinct points are not distinct, then there are the following possibilities.

If $\lambda = 1/\lambda$ then $\lambda = -1$ (harmonic quadruple) and the six values are $\{-1, 2, 1/2\}$. The same three values are obtained from the cases $\lambda = 1 - \lambda$ and $\lambda = \lambda/(\lambda - 1)$. If $\lambda = (\lambda - 1)/\lambda$ (or $\lambda = 1/(1-\lambda)$) then $\lambda^2 - \lambda + 1 = 0$ which has at most two solutions in $\mathbb{F}$. The two solutions over $\mathbb{C}$ are $\lambda = \frac{1 \pm \sqrt{-3}}{2}$ and in this case the quadruple is called *equinharmonic*.

Projection and perspectivities. Consider two distinct lines L and M in $\mathbb{P}^2$ and a point p not lying on either of these lines.

Lemma 17. *The* projection *δ of L to M from p defines a projectivity between the lines L and M.*

Proof. To prove the lemma it suffices to show that δ preserves the cross ratio. With respect to a frame (p_0, p_1, p_2, p_3), where $p_0 = L \cap M$, $p_1 \in L$, $p_2 \in M$ and $p_4 = p$, the point r with coordinates $(y, 1, 0)$ on L

gets projected onto the point s with coordinates $(1-y,0,1)$. Consider the points $r_0 = p_0$, $r_1 = p_1$, $r_2(1,1,0)$ and r on L. Ignoring the last coordinate on L (it is zero on L), this 4-tuple has cross ratio y. The images of these 4 points under the projection from p give the points (this time ignoring the second coordinate, which is zero on M) $q_0(1,0)$, $q_1(1,1)$, $q_2(0,1)$, and $q_3(1-y,1)$. By the above the cross ratio of (q_0,q_1,q_2,q_3) is equal to $1-(1-y)$ (apply the permutation $(23) \in S_4$), which is y. It follows that the projective preserves the cross ratio, and therefore it is a projectivity. □

Complete quadrilateral theorem. A *complete quadrilateral* Δ in $\mathbb{P}^2$ is obtained from a frame in the dual projective plane, and consists of 4 lines (the *sides* of Δ) no three of which are concurrent. The 6 points obtained by intersecting the sides are called the *vertices of* Δ. The lines passing through opposite vertices are called the *diagonals* (there are 3 of them) of Δ and the intersection of the diagonals are called the diagonal points.

Theorem 18. *If M is a diagonal of a complete quadrangle Γ, containing the two vertices q_1, q_2, and the two diagonal points q_3 and q_4, then (q_1,q_2,q_3,q_4) is harmonic. Conversely every harmonic quadruple of points in $\mathbb{P}^1$ can be obtained in this way.*

Proof. Choose appropriate coordinates and verify that the affine coordinate of q_4 w.r.t. (q_1,q_2,q_3) is -1 for a 4-tuple as described in the statement of the theorem. Alternatively, projecting one diagonal D_1 onto another diagonal D_2 from the two vertices which are not contained in either of these diagonals, shows that the cross ratio of (q_1,q_2,q_3,q_4) is invariant under $(12) \in S_4$, which implies that the 4-tuple is harmonic.

Conversely suppose (q_1,q_2,q_3,q_4) is harmonic. Then since $\mathrm{PGL}(2,\mathbb{F})$ preserves the cross-ratio we may assume $q_1(0,1)$, $q_2(1,0)$, $q_3(1,1)$, and $q_4(-1,1)$. Now embed $\mathbb{P}^1$ as the line $X_2 = 0$ in $\mathbb{P}^2$ and find a complete quadrangle satisfying the required properties. □

Lecture 5. The projective groups

Perspectivities. Perspectivities from an important type of collineations. A collineation α of $\mathbb{P}^n$ with $n \geq 2$ is called an *axial collineation* if there exists a hyperplane H such that α fixes each point of H. In this case H is called the *axis* of α. A collineation is α is called a *central collineation* if there exists a point p such that α fixes each hyperplane through p. The point p is called the *centre* of α.

Theorem 19. *If a collineation α of $\mathbb{P}^n$, $n \geq 2$, has an axis, then it also has a centre, and conversely.*

Proof. We show that an axis implies a centre. If α is the identity then there is nothing to prove. So suppose $\alpha \neq id$. Consider the two cases: (i) α has no fixpoints outside its axis; (ii) α has a fixpoint outside its axis. (i) Since α fixes each point of its axis, a hyperplane H, it fixes each subspace contained in H. Consider the line $\ell = rr^\alpha$ where r is point not contained in H. (Such a point exists since $\alpha \neq id$.) Then ℓ intersects the H in a point $p = p^\alpha$, and it follows that ℓ is fixed by α since $\ell^\alpha = (rp)^\alpha = r^\alpha p$. We will show that p is a centre for α. Let m be any other line through p, not contained in H. The plane $\pi = \langle \ell, m \rangle$ meets H in a line ℓ_1 and therefore $\pi = \langle \ell_1, \ell \rangle$. Moreover, since both ℓ and ℓ_1 are fixed by α, also π must be fixed by α. It follows that m^α is contained in π too. Let s be a point on m different from p, i.e. $m = sp$. Consider the line $m_1 = ss^\alpha$, and note that m_1 is contained in π. Using the same argument as was used to prove that $\ell^\alpha = \ell$ one sees that $m_1^\alpha = m_1$. Since m_1 and ℓ belong to π and are both fixed by α, they intersect in point and that point must also be fixed by α. This implies that m_1 meets ℓ in the point p (there are no fixpoints outside H), and it follows that $m = m_1$ which is a line fixed by α. We have shown that each line through p is fixed. But then also each hyperplane through p is fixed by α. We have shown that p is a centre for α.

(ii) Let p be a fixpoint of α with $p \notin H$. Consider any line ℓ through p. Then ℓ meets H in a point r and since both r and p are fixed by α, it follows that also ℓ is fixed by α. This shows that each line through p is fixed by α, i.e. the point p is a centre for α.

For the converse, apply the principle of duality. □

A central (or axial) collineation is also called a *perspectivity*. The group of all perspectivities with common centre p and common axis H is called a *perspectivity group* and is denoted by $\mathrm{Persp}(p, H)$. Perspectivities with incident (centre,axis)-pair are called *elations*, and perspectivities of which the centre is not contained in the axis are called *homologies*.

Theorem 20. *A perspectivity $\alpha \neq id$ of $\mathbb{P}^n$, $n \geq 2$, has a unique axis and a unique centre.*

Proof. Suppose α has two distinct centres p and p'. Consider a point s not on pp' and the lines $\ell = sp$ and $\ell' = sp'$. Then both lines ℓ and ℓ' are fixed by α and therefore s^α must be contained in their intersection, i.e. $s = s^\alpha$. It then easily follows that $\alpha = id$ (convince yourself that α also fixes every point on pp'). The uniqueness of the axis follows from the principle of duality. □

Theorem 21. *Given a point-hyperplane pair (p, H) and two points $x \neq x'$ of $\mathbb{P}^n$, $n \geq 2$, not contained in $H \cup \{p\}$, with x, x', p collinear, there exists a unique $\alpha \in \mathrm{Persp}(p, H)$ such that $x^\alpha = x'$.*

Proof. Suppose $\dim V = n + 1$. First we prove uniqueness. If $x^\alpha = x'$ and $x^\beta = x'$ with $\alpha, \beta \in \mathrm{Persp}(p, H)$, then $x^{\alpha\beta^{-1}} = x$ with $\alpha\beta^{-1} \in \mathrm{Persp}(p, H)$. Consider any hyperplane H' through x. Since $x \notin H$ the hyperplane $H' = \langle x, H \cap H' \rangle$, and it follows that H' is also fixed by $\alpha\beta^{-1}$. This shows that $x \neq p$ is also centre of $\alpha\beta^{-1}$. By the uniqueness of the centre (Theorem 21) it follows that $\alpha\beta^{-1} = id$.

In order to show the existence of $\alpha \in \mathrm{Persp}(p, H)$ such that $x^\alpha = x'$, we distinguish between elations and homologies, i.e. $p \in H$ and $p \notin H$.

(a) If $p \in H$ then consider any point $y \notin H$, and two frames

$$\Lambda(p, p_1, \ldots, p_{n-1}, x, y) \text{ and } \Lambda'(p, p_1, \ldots, p_{n-1}, x', y'),$$

where $z = xy \cap H$, $y' = py \cap x'z$, and $p_1, \ldots, p_{n-1} \in H$. Then $\Sigma_a(p, p_1, \ldots, p_{n-1}, z)$ is a frame of H.

(b) If $p \notin H$ then consider two frames

$$\Lambda(p, p_1, \dots, p_n, x) \text{ and } \Lambda'(p, p_1, \dots, p_n, x'),$$

where $p_1, \dots, p_n \in H$. Then $\Sigma_b(p_1, \dots, p_n, z)$, with $z = px \cap H$ is a frame of H.

In both cases (a) and (b) let α be the unique projectivity satisfying $\Lambda^\alpha = \Lambda'$. Then α fixes both frames Σ_a and Σ_b, and this implies that H is an axis of α and is therefore a perspectivity.

It follows that α also has a centre and we are left to show that the centre of α is p. This immediately follows in case (b) since α fixes p and $p \notin H$.

In case (a), suppose p' is the centre of α. Clearly $p' \notin \{x, x', y, y'\}$. Then the line xp' is fixed by α, imlying $p' \in xx'$. The same argument using the points y and y' shows that p' is on the line yy'. It follows that $p = p'$. $\square$

Remark 22. *Choosing a convenient frame in part* (a) *of the above proof (precisely* $p = \langle e_1 \rangle$, $H = \langle e_1 \dots, e_n \rangle$, $x(1, 0, \dots, 0)$ *and* $x'(1, a, 0, \dots, 0)$*), we see that the elation* α *is induced by* $\varphi = (A, id)$*, with*

$$A = \left(\begin{array}{cc|c} 1 & 0 & \\ a & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-1} \end{array}\right).$$

Theorem 23. *If* α *is an elation of* $\mathrm{PG}(V)$*, with* $\dim V \geq 3$*, then* $\alpha \in \mathrm{PSL}(V)$*.*

Proof. The above remark combined with the fact that $\mathrm{PGL}(V)$ acts transitively on frames and $\det(g^{-1}Ag) = \det(A)$. $\square$

Analogously to the definition of collineations of a projective line, the previous Corollary motivates the following definition of elations on a projective line. An *elation of* $\mathbb{P}^1$ is a projectivity $\alpha = \overline{(A, id)}$ with 1 fixpoint and such A has eigenvalue 1 with algebraic multiplicity 2.

The fundamental diagram. Our next aim is to understand the following diagram of exact sequences.

$$\begin{tikzcd}
\mathrm{SL}(V) \cap \mathrm{Sc}(V) \arrow[r] \arrow[d] & \mathrm{Sc}(V) \arrow[r, "det"] \arrow[d] & N \arrow[d] \\
\mathrm{SL}(V) \arrow[r] \arrow[d, "\varphi \mapsto \bar{\varphi}"] & \mathrm{GL}(V) \arrow[r, "det"] \arrow[d, "\varphi \mapsto \bar{\varphi}"] & \mathbb{F}^* \arrow[d, "a \mapsto a^{n+1}"] \\
\mathrm{PSL}(V) \arrow[r] & \mathrm{PGL}(V) \arrow[r] & \mathbb{F}^*/N
\end{tikzcd}$$

FIGURE 1. A diagram of exact sequences with $V = \mathbb{F}^{n+1}$ and $N := \{a^{n+1} : a \in \mathbb{F}^*\}$.