Homework on Ideals

Let R be a commutative ring.

For two ideals I and J of R, we let,

$$I + J = \{i + j : i \in I \text{ and } j \in J\}$$

and

$$IJ = \text{set of finite sums of elements of the form } ij \text{ for } i \in I \text{ and } j \in J.$$

1. Show that for $I, J \lhd R$, we have $I + J$, $IJ \lhd R$. Show that $IJ \subseteq I \cap J$. Show that $I + J$ is the smallest ideal containing $I \cup J$. Show that for IJ to be an ideal of R it is enough that only one of I and J is an ideal of R. Give an example where $IJ = I \cap J$ and an example where $IJ \neq I \cap J$.

2. Show that for $I, J, K \lhd R$ if $I \subseteq J$ then $J \cap (I + K) = I + (J \cap K)$.

3. Show that for $I, J, K \lhd R$ we have $I(J + K) = IJ + IK$.

4. Define the sum and the product of any set of ideals. Defining I^n as the product of I with itself n times, for each n find an example where $I^n = 0$ but $I^{n-1} \neq 0$. Find an example where $\cap_n I^n \neq 0$

For two ideals I and J of R, define,

$$[I : J] = \{r \in R : rJ \subseteq I\}.$$

5. Show that for $I, J \lhd R$, $[I : J]$ is also an ideal of R containing I.

6. Show that for $I \lhd R$, $[I : R] = I$.

7. Show that for $I, J \lhd R$, $J \subseteq I$ iff $[I : J] = R$.

8. For ideals of R show that $\left[\bigcap_{k=1}^{n} I_k : J\right] = \bigcap_{k=1}^{n} [I_k : J]$.

9. For ideals of R show that $\left[I : \sum_{k=1}^{n} J_k\right] = \bigcap_{k=1}^{n} [I : J_k]$.

10. Show that for $I, J, K \lhd R$ we have $[I : JK] = [[I : J] : K]$.

For an ideal $I \lhd R$, define $\sqrt{I} = \{r \in R : r^n \in I\}$ = radical of I.

11. Show that for $I \lhd R$, $\sqrt{I} \lhd R$.

12. Show that for $I, J \lhd R$, if $I^k \subseteq J$ for some natural number k then $\sqrt{I} \subseteq \sqrt{J}$.

13. Show that for $I, J \lhd R$, $\sqrt{(IJ)} = \sqrt{(I \cap J)} = \sqrt{I} \cap \sqrt{J}$.

14. Show that for $I, J \lhd R$, and that $\sqrt{\sqrt{I}} = \sqrt{I}$.