Homework on Ideals

Let $R$ be a commutative ring.

For two ideals $I$ and $J$ of $R$, we let,

$I + J = \{ i + j : i \in I \text{ and } j \in J \}$

and

$IJ = \text{set of finite sums of elements of the form } ij \text{ for } i \in I \text{ and } j \in J$.

1. Show that for $I, J \triangleleft R$, we have $I + J, IJ \triangleleft R$. Show that $I + J$ is the smallest ideal containing $I \cup J$. Show that for $IJ$ to be an ideal of $R$ it is enough that only one of $I$ and $J$ is an ideal of $R$. Give an example where $IJ = I \cap J$ and an example where $IJ \neq I \cap J$.

2. Show that for $I, J, K \triangleleft R$ if $I \subseteq J$ then $J \cap (I + K) = I + (J \cap K)$.

3. Show that for $I, J, K \triangleleft R$ we have $I(J + K) = IJ + IK$.

4. Define the sum and the product of any set of ideals. Defining $I^n$ as the product of $I$ with itself $n$ times, for each $n$ find an example where $I^n = 0$ but $I^{n-1} \neq 0$. Find an example where $\cap_n I^n \neq 0$

5. For two ideals $I$ and $J$ of $R$, define,

$[I : J] = \{ r \in R : rJ \subseteq I \}$.

6. Show that for $I \triangleleft R, [I : J] = I$.

7. Show that for $I \triangleleft R, J \subseteq I$ iff $[I : J] = R$.

8. For ideals of $R$ show that $\bigcap_{k=1}^n I_k : J = \bigcap_{k=1}^n [I_k : J]$.

9. For ideals of $R$ show that $\bigcap_{i=1}^n \bigcup_{j=1}^n I_k : J_k = \bigcup_{k=1}^n [I : J_k]$.


For an ideal $I \triangleleft R$, define $\sqrt{I} = \{ r \in R : r^n \in I \}$ = radical of $I$.

11. Show that for $I \triangleleft R, \sqrt{I} \triangleleft R$.

12. Show that for $I \triangleleft R$, if $I^k \subseteq J$ for some natural number $k$ then $\sqrt{I} \subseteq \sqrt{J}$.

13. Show that for $I, J \triangleleft R$, $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

14. Show that for $I, J \triangleleft R$, and that $\sqrt[\infty]{I} = \sqrt{I}$.