## Homework on Ideals

Let $R$ be a commutative ring.
For two ideals $I$ and $J$ of $R$, we let,

$$
I+J=\{i+j: i \in I \text { and } j \in J\}
$$

and

$$
I J=\text { set of finite sums of elements of the form } i j \text { for } i \in I \text { and } j \in J .
$$

1. Show that for $I, J \triangleleft R$, we have $I+J, I J \triangleleft R$. Show that $I J \subseteq I \cap J$. Show that $I+J$ is the smallest ideal containing $I \cup J$. Show that for $I J$ to be an ideal of $R$ it is enough that only one of $I$ and $J$ is an ideal of $R$. Give an example where $I J=I \cap J$ and an example where $I J \neq I \cap J$.
2. Show that for $I, J, K \triangleleft R$ if $I \subseteq J$ then $J \cap(I+K)=I+(J \cap K)$.
3. Show that for $I, J, K \triangleleft R$ we have $I(J+K)=I J+I K$.
4. Define the sum and the product of any set of ideals. Defining $I^{n}$ as the product of $I$ with itself $n$ times, for each $n$ find an example where $I^{n}=0$ but $I^{n-1} \neq 0$. Find an example where $\cap_{n} I^{n}$ $\neq 0$

For two ideals $I$ and $J$ of $R$, define,

$$
[I: J]=\{r \in R: r J \subseteq I\} .
$$

5. Show that for $I, J \triangleleft R,[I: J]$ is also an ideal of $R$ containing $I$.
6. Show that for $I \triangleleft R,[I: R]=I$.
7. Show that for $I, J \triangleleft R, J \subseteq I$ iff $[I: J]=R$.
8. For ideals of $R$ show that $\left.\bigcap_{k=1}^{n} I_{k}: J\right]=\bigcap_{k=1}^{n}\left[I_{k}: J\right]$.
9. For ideals of $R$ show that $\left\lfloor I: \sum_{k=1}^{n} J_{k}\right\rfloor=\bigcap_{k=1}^{n}\left[I: J_{k}\right]$.
10. Show that for $I, J, K \triangleleft R$ we have $[I: J K]=[[I: J]: K]$.

For an ideal $I \triangleleft R$, define $\sqrt{ } I=\left\{r \in R: r^{n} \in I\right\}=$ radical of $I$.
11. Show that for $I \triangleleft R, \sqrt{ } I \triangleleft R$.
12. Show that for $I, J \triangleleft R$, if $I^{k} \subseteq J$ for some natural number $k$ then $\sqrt{ } I \subseteq \sqrt{ } J$.
13. Show that for $I, J \triangleleft R, \sqrt{ }(I J)=\sqrt{ }(I \cap J)=\sqrt{ } I \cap \sqrt{ } J$.
14. Show that for $I, J \triangleleft R$, and that $\sqrt{ } \sqrt{ } I=\sqrt{ } I$.

