Homework on Ideals

Let *R* be a commutative ring.

For two ideals I and J of R, we let,

$$I + J = \{i + j : i \in I \text{ and } j \in J\}$$

and

IJ = set of finite sums of elements of the form ij for $i \in I$ and $j \in J$.

- **1.** Show that for $I, J \triangleleft R$, we have $I + J, IJ \triangleleft R$. Show that $IJ \subseteq I \cap J$. Show that I + J is the smallest ideal containing $I \cup J$. Show that for IJ to be an ideal of R it is enough that only one of I and J is an ideal of R. Give an example where $IJ = I \cap J$ and an example where $IJ \neq I \cap J$.
 - **2.** Show that for $I, J, K \triangleleft R$ if $I \subseteq J$ then $J \cap (I + K) = I + (J \cap K)$.
 - **3.** Show that for I, J, $K \triangleleft R$ we have I(J + K) = IJ + IK.
- **4.** Define the sum and the product of any set of ideals. Defining I^n as the product of I with itself n times, for each n find an example where $I^n = 0$ but $I^{n-1} \neq 0$. Find an example where $\bigcap_n I^n \neq 0$

For two ideals I and J of R, define,

$$[I:J] = \{ r \in R : rJ \subseteq I \}.$$

- **5.** Show that for $I, J \triangleleft R$, [I : J] is also an ideal of R containing I.
- **6.** Show that for $I \triangleleft R$, [I:R] = I.
- **7.** Show that for $I, J \triangleleft R, J \subseteq I$ iff [I : J] = R.
- **8.** For ideals of R show that $\left[\bigcap_{k=1}^{n} I_k : J\right] = \bigcap_{k=1}^{n} [I_k : J]$.
- **9.** For ideals of *R* show that $\left[I:\sum_{k=1}^n J_k\right] = \bigcap_{k=1}^n \left[I:J_k\right]$.
- **10.** Show that for I, J, $K \triangleleft R$ we have [I:JK] = [[I:J]:K].

For an ideal $I \triangleleft R$, define $\sqrt{I} = \{r \in R : r^n \in I\} = \text{radical of } I$.

- **11.** Show that for $I \triangleleft R$, $\sqrt{I} \triangleleft R$.
- **12.** Show that for $I, J \triangleleft R$, if $I^k \subseteq J$ for some natural number k then $\sqrt{I} \subseteq \sqrt{J}$.
- **13.** Show that for $I, J \triangleleft R, \sqrt{(IJ)} = \sqrt{(I \cap J)} = \sqrt{I} \cap \sqrt{J}$.
- **14.** Show that for $I, J \triangleleft R$, and that $\sqrt{\sqrt{I}} = \sqrt{I}$.