Math 212 (Abstract Algebra)
MT 2008
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I. We consider the $\mathbb{C}$-algebra $M = \mathrm{Mat}_{2\times 2}(\mathbb{C})$.

1. Show that any 2×2 matrix over $\mathbb{C}$ is conjugate to one of the following two types of matrices (20 pts.):
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$
2. Show that for any $m \in M$ there is a nonzero monic polynomial $p_m(X) \in \mathbb{C}[X]$ of degree ≤ 4 such that $p_m(m) = 0$. (5 pts.)
3. Show that the monic polynomial $p_m(X)$ above can be chosen so that for all $q(X) \in M$, $p_m(\mathrm{X})$ divides $q(X)$ whenever $q(m) = 0$. ($p_m(X)$ is called the minimal polynomial of m). (5 pts.)
4. Show that the subalgebra $\mathbb{C}[m]$ of M generated by $\mathbb{C}$ (view $\mathbb{C}$ as the subalgebra of M that consists of scalar matrices) and m is isomorphic to $\mathbb{C}[X]/\langle p_m(X)\rangle$. (Here p_m is chosen as in Q3). (5 pts.)
5. Let $m \in M$, $g \in \mathrm{GL}_2(\mathbb{C})$ and $p(X) \in \mathbb{C}[X]$. Show that if $p(m) = 0$ then $p(g^{-1}mg) = 0$. (5 pts.)
6. Show that for any $m \in M$ there is a nonzero polynomial $p_m(X) \in \mathbb{C}[X]$ of degree ≤ 2 such that $p_m(m) = 0$. Find examples where $\deg p_m = 1$ and $\deg p_m > 1$. (5 pts.)
7. Show that $\mathbb{C}[m]$ is isomorphic to one of the three types of $\mathbb{C}$-algebras: $\mathbb{C}$, $\mathbb{C}[X]/X^2$ and $\mathbb{C}[X]/\langle X^2 - cX\rangle$ for $c \in \mathbb{C}$. (10 pts.)
8. Are any of the above $\mathbb{C}$-algebras isomorphic to each other? (10 pts.)

II. Now we consider the $\mathbb{R}$-algebra $M = \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

1. Let $\rho(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $\theta \in [0, 2\pi)$. Show that $\rho(\theta)$ is the anti-clockwise rotation of angle θ around the origine. (5 pts.)
2. Show that there is a polynomial $p_\theta(X) \in \mathbb{R}[X]$ of degree ≤ 2 such that $p_\theta(\rho(\theta)) = 0$. (10 pts.) Find all θ for which the polynomial $p_\theta(X)$ is irreducible of degree 2. (5 pts.)
3. Suppose $p_\theta(X)$ is irreducible of degree 2. Show that the $\mathbb{R}$-algebra $\mathbb{R}[\rho(\theta)]$ generated by $\rho(\theta)$ is isomorphic to the field $\mathbb{C}$ of complex numbers. (5 pts.)
4. Show that any $m \in M$ has a minimal polynomial of degree ≤ 2 over $\mathbb{R}$. (20 pts.)

III. For $m \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$, study the isomorphism type of the $\mathbb{Z}$-module $\mathbb{Z}[m]$ generated by ring of the scalar matrices ($\approx \mathbb{Z}$) and the matrix m. (50 pts.)