Math 211 (Abstract Algebra)

Quiz and Homework (on localization)
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Let R be a commutative ring with 1. Let S be a multiplicatively closed subset of R containing 1. On $R \times S$ define the relation,

$$(r, s) \approx (r', s')$$

by the rule

"There exists an $s_1 \in S$ such that $s_1(s'r - ra') = 0$".

1. Show that this is an equivalence relation.

Let $S^{-1}R = (R \times S)/\approx$. Denote the equivalence class of (r, s) by the suggestive notation r/s.

2. Note that if $0 \in S$, then $S^{-1}R$ has only one element. From now on we assume that $0 \notin S$.

3. Show that on $S^{-1}R$ one can define addition and multiplication. With these operations $S^{-1}R$ becomes a ring (called the **ring of fractions of *R* by *S***).

4. Show that there is a natural ring homomorphism φ from R into $S^{-1}R$. When is it an embedding?

5. Show that the elements of $\varphi(S)$ are invertible in $S^{-1}R$.

6. Show that if $S \subseteq R^*$, then $S^{-1}R \approx R$.

7. Let T be a ring and α: $R \to T$ be a ring homomorphism such that $\alpha(S)$ is invertible in T. Show that there is a unique ring homomorphism β: $S^{-1}R \to T$ such that $\beta \circ \varphi = \alpha$.

8 and 9. Let P be a prime ideal of R (i.e. $xy \in P$ implies that either x or y is in P). Let $S = R \setminus P$. Show that S is a multiplicatively closed subset of R containing 1. Show that $S^{-1}R$ is a local ring (i.e. it has a unique maximal ideal, or equivalently the set of noninvertible elements of it forms an ideal; prove that assertion). $S^{-1}R$ is called the **local ring of *R* at *P*.**

10. Conclude that if R has no zerodivisors and $S = R \setminus \{0\}$, then $S^{-1}R$ is a field (called the **quotient field** or the **field of fractions of *R***).

Note that if $R = \mathbb{Z}$, then $S^{-1}R \approx \mathbb{Q}$.

If $R = F[X]$ is the ring of polynomials over the field F, then $S^{-1}R$ is the field of fractional polynomials over F.

11. Let $J(R)$ denote the set of ideals of a ring R. Let ψ: $J(R) \to J(S^{-1}R)$ be defined by $\psi(I) = \varphi(I) = S^{-1}I$. Show that

11a. $\psi(I + J) = \psi(I) + \psi(J)$

11b. $\psi(I\,J) = \psi(I)\,\psi(J)$

11c. $\psi(I \cap J) = \psi(I) \cap \psi(J)$

12. Is ψ onto?