Lie Algebras HW9

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Let $L = \mathrm{sl}_2(F)$ where F is algebraically closed. Let

$$x = \begin{pmatrix} 0\,1 \\ 0\,0 \end{pmatrix},\ y = \begin{pmatrix} 0\,0 \\ 1\,0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the standard basis of $\mathrm{sl}_2(F)$ (regarded as a vector space).

We will classify irreducible finite dimensional L-modules. Let V be such a module. For $\lambda \in F$, set

$$V_\lambda = \{v \in V : hv = \lambda v\}.$$

V_λ is a subspace of V. Note that λ is an eigen-value iff $V_\lambda \neq 0$. In this case we call λ a **weight** and V_λ a **weight space**.

1. Check that $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$.

2. Show that V is a direct sum of distinct weight spaces.

3. Show that $xV_\lambda \leq V_{\lambda+2}$ and that $yV_\lambda \leq V_{\lambda-2}$. Conclude that there is a weight space V_λ such that $xV_\lambda = 0$.

A nonzero vector of a weight space V_λ such as above is called a **maximal vector** of weight λ. We choose a maximal vector $v_o \in V_{gl}$ Let $v_{-1} = 0$ and for $i \geq 0$, define

$$v_i = (1/i!)\, y^i v_o.$$

4. Show that for $i \geq 0$, $hv_i = (\lambda-2i)v_i$, i.e. $v_i \in V_{\lambda-2i}$.

5. Show that for $i \geq 0$, $yv_i = (i + 1)v_{i+1}$.

6. Show that for $i \geq 0$, $xv_i = (\lambda-i+1)v_{i-1}$.

7. Show that $\mathrm{Vect}(v_i : i \geq 0) = \oplus_{i \geq 0} Fv_i$ and is an L-module.

9. Show that $\mathrm{Vect}(v_i : i \geq 0) = L$ and that if $V_\lambda \neq 0$ then $V_\lambda = \mathrm{Vect}(v$.

11. Let $m = \dim V$. Show that $v_o \in V_m$.

12. Show that either all the weights λ are even or they are all odd.

13. Show that

4. Show that