Lie Algebras

Midterm I
Gümüşlük Akademisi
Ali Nesin
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Recall

i. Let L be a Lie algebra over a field F. An ***L*-module** V is a vector space over F together with a bilinear map $L \times V \to V$ that sends the pair $(l, v) \in L \times V$ to an element denoted $lv \in V$ such that $[l, l']v = l(l'v) - l'(lv)$ for all $l, l' \in L$ and $v \in V$.

ii. If V and W are vector spaces with bases $(v_i)_{i\in I}$ and $(w_j)_{j\in J}$, then the tensor product $V \otimes W$ has basis $(v_i \otimes w_j)_{i\in I,\ j\in J}$.

iii. If V is a vector space, V^* denotes the vector space $\mathrm{Hom}_F(V, F)$ of linear maps from V into F.

All vector spaces and Lie algebras we consider are assumed to be finite dimensional over the base field *F*.

1. (Dual Basis). Let $f : V \times V \to F$ be a symmetric nondegenerate bilinear form. Let $v_1, ..., v_n$ be a basis of V. Show that there is a unique basis (called the **dual** base) $w_1, ..., w_n$ such that $f(v_i, w_j) = \delta_{ij}$. (20 pts.)

2. Show that an L-module V is a direct sum of irreducible modules iff it is completely reducible (i.e. its submodules split, i.e. its submodules have complements). (7 pts.)

3. (Dual Space). Let V be an L-module. Let V^* be the dual space of V (considered as a vector space). For $l \in L$ and $f \in V^*$, define $lf : V \to F$ by $(lf)(v) = -f(lv)$ for all $v \in V$. Show that $lf \in V^*$ and that this multiplication defines a Lie module structure on V^*. (5 pts.)

4. Let V and W be vector spaces.

4a. Show that there is a unique linear map φ from $V^* \otimes W$ into $\mathrm{Hom}_F(V, W)$ such that $\varphi(f \otimes w)(v) = f(v)w$ for all $f \in V^*$, $w \in W$ and $v \in V$. (6 pts.)

4b. Let $v_1, ..., v_n$ and $w_1, ..., w_m$ be bases of V and W respectively. Compute the endomorphism $\varphi(v_i^* \otimes w_j)$. (4 pts.)

4c. Deduce that φ is an isomorphism of vector spaces. (5 pts.)

5. (Tensor Product of Modules). Let V and W be L-modules. For $l \in L$, $v \in V$ and $w \in W$ define $l(v \otimes w) = lv \otimes w + v \otimes lw$. Show that this **defines** a Lie algebra structure on $V \otimes W$. (6 pts.)

6. Let V and W be L-modules. By #3, V^* has an L-module structure. By #5, $V^* \otimes W$ has an L-module structure. By #4, this L-module structure can be transferred to $\mathrm{Hom}_F(V, W)$ via $l\alpha = \varphi(l\, \varphi^{-1}(\alpha))$. (Here $l \in L$, $\alpha \in \mathrm{Hom}_F(V, W)$, φ is as in # 4).

Check that for $l \in L$ and $\alpha \in \mathrm{Hom}_F(V, W)$, $l\alpha$ is given by $(l\alpha)(v) = l\alpha(v) - \alpha(lv)$ for $v \in V$. (7 pts.)

7. (Casimir Element). Let L be a semisimple Lie algebra over an algebraically closed field of characteristic 0 and let φ: $L \to \mathrm{gl}(V)$ be a faithful representation of L (i.e. φ is a one-to-one Lie algebra homomorphism). For $x, y \in L$, define $\beta(x, y) = \mathrm{tr}(\varphi(x)\varphi(y))$.

7a. Show that β is a symmetric associative nondegenerate bilinear form. (5 pts.)

Let $x_1, ..., x_n$ be a basis of L. By #1, there is a dual basis $y_1, ..., y_n$ such that $\beta(x_i, y_j) = \delta_{ij}$ for all i, j. For $x\in L$, let $a_{ij}(x)$ and $b_{ij}(x) \in F$ be such that for all i, j,

$$[x, x_i] = \Sigma_j\, a_{ij}(x)\, x_j$$

$$[x, y_i] = \Sigma_j\, b_{ij}(x)\, y_j$$

7b. Show that $\beta([x, x_i], y_k) = -\beta(x_i, [x, y_k])$. By using that equality show that $a_{ik}(x) = -b_{ki}(x)$. (5 pts.)

Let $c = \Sigma_i\, \varphi(x_i)\varphi(y_i) \in \mathrm{gl}(V)$. c is called **Casimir element** of φ.

7d. Show that tr(c) = dim L. (5 pts.)

7c. Show that $[\varphi(x), c] = 0$ for all $x\in L$. Thus $c \in C_{\mathrm{End}(V)}(\varphi(L))$. (10 pts.)

7d. Assume φ is irreducible. Show that $c = (\dim L/\dim V)$ Id. In particular c is independent of the choice of the basis $x_1, ..., x_n$. (5 pts.)

7e. Compute c in case $L = \mathrm{sl}_2(F)$, $V = F^2$ and φ = Id by going through the definitions. (10 pts.)