Lie Algebras HW7

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Let L be a finite dimensional Lie algebra. Let $\kappa : L \times L \to F$ be defined by

$$\kappa(x, y) = \mathrm{tr}((\mathrm{ad}\ x)(\ \mathrm{ad}\ y)).$$

κ is a symmetric bilinear form called the **Killing form** of L.

1. Show that the Killing form is associative in the sense that for all $x, y, z \in L$, $\kappa([x, y], z] = \kappa(x, [y, z])$.

2. Compute the Killing form of the Lie algebra of 3×3 strictly upper matrices.

3. Compute the Killing form of $\mathrm{sl}_2(F)$.

4. Lie's Theorem states that over an algebraically closed field **F**, a finite dimensional solvable Lie algebra has a common eigenvector in case char(**F**) = 0 or dim(L) < char(**F**). Here we find a counterexample in case dim(L) = char(**F**).

Let **F** be any field of characteristic $p > 0$. Let V be a vector space of dimension p. Let $v_0, \ldots, v_{p-1}$ be a basis of V. Define $x, y \in \mathrm{gl}(V)$ by

$$x(v_i) = v_{i-1} \text{ (we set } v_{-1} = v_{n-1})$$
$$y(v_i) = iv_i$$

for all $i = 0, 1, \ldots, p - 1$.

Let A be the sub Lie algebra of gl(V) generated by x and y.

4a. Show A is solvable

4b. Show that A has no common eigenvector.

4c. Show that A' has nonnilpotent endomorphisms if $p = 2$. Is this true if $p > 2$?

5. We will now find a counterexample in all characteristics $p > 0$ to a statement that we know it holds in characteristic 0: *If L is solvable then L′ is nilpotent.*

Let A and F be as above. Let $B = A \oplus \mathbf{F}^p$ (direct sum as a vector space). Turn B into a Lie algebra by decreeing that $\mathbf{F}^p$ is abelian, that A acts on $\mathbf{F}^p$ as usual and that A has its Lie product.

5a. Check that this really gives a Lie algebra structure to B.

5b. Check that $B' = \mathbf{F}x \oplus \mathbf{F}^p$.

5c. Check that B is solvable.

5d. Check that B' is not nilpotent.