Lie Algebras HW6

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1. Let F be a field and $G = \mathrm{GL}_n(F)$. Let $D_n(F)$ and $B_n(F)$ be the set of diagonal and upper triangular matrices of G.

1a. Show that $\mathrm{N}_G(D_n(F))/D_n(F) \approx \mathrm{Sym}(n)$.

1b. Show that $D_n(F)$ splits in $\mathrm{N}_G(D_n(F))$, i.e. show that there is a subgroup $S \leq \mathrm{N}_G(D_n(F))$ such that $\mathrm{N}_G(D_n(F)) = D_n(F)S$ and $D_n(F) \cap S = 1$.

1c. Show that $\mathrm{N}_G(B_n(F)) = B_n(F)$

2. Let F be a field and $L = \mathrm{gl}_n(F)$. Let $d_n(F)$ and $b_n(F)$ be the set of diagonal and upper triangular matrices of L. Assume $\mathrm{char}(F) = 0$

2a. Show that $\mathrm{N}_L(d_n(F)) = d_n(F)$.

2b. Show that $\mathrm{N}_L(b_n(F)) = b_n(F)$.

2c. What happens in characteristic $p > 0$?

3. Show that if $\mathrm{char}(F) = 2$ then $\mathrm{sl}_2(F)$ is nilpotent.

4. Show that if $\mathrm{char}(F) = p > 0$ then $\mathrm{sl}_p(F)$ is not simple.

5. Let L be a nilpotent Lie algebra. Let K be an ideal of codimension 1 (why does it have to exist?)

5a. Show that $\mathrm{C}_L(K) \neq 0$.

5b. Let $x \in L \setminus K$. Let n be the smallest integer such that $\mathrm{C}_L(K) \leq L^n$. Let $z \in \mathrm{C}_L(K) \setminus L^{n+1}$. Let $\delta : L \to L$ be the linear map that sends K to 0 and x to z. Show that δ is a derivation.

5c. Show that for all $l \in L$, $\delta \neq \mathrm{ad}(l)$.

6. Let L be a finite dimensional Lie algebra, $K \triangleleft L$ be such that L/K is nilpotent and such that $(\mathrm{ad}\, x)|_K$ is nilpotent for all $x \in L$. Show that L is nilpotent.

7a. Show that $\mathrm{Z}(\mathrm{gl}_n(F)) = F\mathrm{Id}$.

7b. Show that if F is algebraically closed, then $\mathrm{gl}_n(F) = \mathrm{sl}_n(F).\mathrm{Z}(\mathrm{gl}_n(F))$.

7c. Find $\mathrm{Z}(\mathrm{gl}_n(F))$.

7d. Use Lie's theorem to prove that if $\mathrm{char}(F) = 0$ or if $n < \mathrm{char}(F)$ then $\mathrm{Rad}(\mathrm{sl}_n(F)) = \mathrm{Z}(\mathrm{sl}_n(F))$.