Lie Algebras HW5

Gümüşlük Akademisi
Ali Nesin
July 28rd, 2000

1. Let L be a 3 dimensional Lie Algebra over a field K. Let e_1, e_2, e_3 be a basis of L. For $i, j = 1, 2, 3$ and $i < j$ set

$$[e_i, e_j] = a_{ij1}e_1 + a_{ij2}e_2 + a_{ij3}e_3. \qquad (*)$$

1a. Show that the constants $(a_{i,j,k})_{i<j,k}$ determine the Lie algebra structure of L.

1b. What are the relations between the constants $(a_{i,j,k})_{i<j,k}$ for the multiplication (*) to define a Lie algebra structure?

2. Show that a Lie algebra L where $Z(L)$ has codimension ≤ 1 in L is abelian.

3. Show that a Lie algebra L where $Z(L)$ has codimension ≤ 2 in L is solvable.

4. Show that a nilpotent Lie algebra of dimension 2 is abelian.

5. Show that a nilpotent Lie algebra of dimension 3 is nilpotent of class ≤ 2.

6. Classify all nilpotent Lie algebras of dimension 3.

7. Show that a nilpotent Lie algebra over F of dimension 3 is a subalgebra of $\mathrm{gl}_n(F)$.

8. Show that a Lie algebra over F of dimension 3 is a subalgebra of some $\mathrm{gl}_n(F)$.

9. Let A be an algebra (not necessarily associative) over a field F. Let $\delta \in \mathrm{Der}(A)$.

9a. Show that for $\lambda, \mu \in F$ and $x, y \in A$,

$$(\delta - (\lambda + \mu)\mathrm{Id})^n(xy) = \Sigma_{i=0,1,\ldots,n} ((\delta - \lambda\mathrm{Id})^{n-i}x)((\delta - \mu\mathrm{Id})^{i}y).$$

9b. For $\lambda \in F$, let $A_\lambda = \{x \in A : (\delta - \lambda\mathrm{Id})^k(x) = 0 \text{ for some } k \in \mathbf{N}\}$. Show that $A_\lambda A_\mu \leq A_{\lambda+\mu}$..