1. Let $L$ be a 3 dimensional Lie Algebra over a field $K$. Let $e_1, e_2, e_3$ be a basis of $L$. For $i, j = 1, 2, 3$ and $i < j$ set

$$[e_i, e_j] = a_{ij_1}e_1 + a_{ij_2}e_2 + a_{ij_3}e_3. \tag{\ast}$$

1a. Show that the constants $(a_{i,j,k})_{i < j, k}$ determine the Lie algebra structure of $L$.

1b. What are the relations between the constants $(a_{i,j,k})_{i < j, k}$ for the multiplication (\ast) to define a Lie algebra structure?

2. Show that a Lie algebra $L$ where $Z(L)$ has codimension $\leq 1$ in $L$ is abelian.

3. Show that a Lie algebra $L$ where $Z(L)$ has codimension $\leq 2$ in $L$ is solvable.

4. Show that a nilpotent Lie algebra of dimension 2 is abelian.

5. Show that a nilpotent Lie algebra of dimension 3 is nilpotent of class $\leq 2$.

6. Classify all nilpotent Lie algebras of dimension 3.

7. Show that a nilpotent Lie algebra over $F$ of dimension 3 is a subalgebra of $\text{gl}_n(F)$.

8. Show that a Lie algebra over $F$ of dimension 3 is a subalgebra of some $\text{gl}_n(F)$.

9. Let $A$ be an algebra (not necessarily associative) over a field $F$. Let $\delta \in \text{Der}(A)$.

9a. Show that for $\lambda, \mu \in F$ and $x, y \in A$,

$$(\delta - (\lambda + \mu)\text{Id})^n(xy) = \sum_{i=0, 1, \ldots, n} (((\delta - \lambda\text{Id})^{n-i}x)(\delta - \mu\text{Id})^i y).$$

9b. For $\lambda \in F$, let $A_\lambda = \{x \in A : (\delta - \lambda\text{Id})^k(x) = 0 \text{ for some } k \in \mathbb{N}\}$. Show that $A_{\lambda_1}A_{\mu} \leq A_{\lambda_1 + \mu}$. 