Lie Algebras HW3

Derivations and Exponentiations

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1. Let V be a vector space and let $\alpha \in \mathrm{End}(V)$ be a **nilpotent** endomorphism of **nilpotency degree** k for some natural numbers $k > 0$, this means that $\alpha^k = 0$ and $\alpha^i \neq 0$ for $i < k$. In case the characteristic p of the base field > 0, we assume that $k \leq p$.

Define (computing in $\mathrm{End}_F(V) = \mathrm{gl}(V)$)

$$\exp \alpha = \mathrm{Id} + \alpha + \alpha^2/2! + ... + \alpha^{k-1}/(k-1)!$$

1a. Show that the vector space endomorphism η of A defined by $\eta = \exp \alpha - \mathrm{Id}$ is nilpotent of nilpotency degree $\leq k$.

1b. Conclude that $\mathrm{Id} - \eta + \eta^2 - ... \pm \eta^{k-1}$ is the inverse of $\exp \alpha$, so that $\exp \alpha \in \mathrm{GL}(V)$.

1c. Assume α and β are commuting nilpotent endomorphisms of nilpotency degree k and l respectively. Show that $\alpha + \beta$ is a nilpotent endomorphism of nilpotency degree $\leq k + l$.

1d. Let α and β be as above. Assume further that either $p = 0$ or $\max(k, l) \leq p$. Show that $\exp(\alpha + \beta) = (\exp \alpha)(\exp \beta)$.

1e. Show that $\exp(\alpha)^{-1} = \exp(-\alpha)$.

1f. Is it possible to define $\exp \alpha$ for nilpotent α without the restriction on the characteristic and the nilpotency class?

1g. Show that every strictly upper triangular $n \times n$ matrix (with zeroes on the diagonai and below) is nilpotent of class $\leq n$.

2. (Leibniz Rule). Let A be an algebra. Let $\delta \in \mathrm{Der}(A)$ be a derivation. Show that for all natural numbers n,

$$\delta^n(xy) = \Sigma_{i=0,\ldots,n} \binom{n}{i} \delta^i(x)\delta^{n-i}(y).$$

3. Let A be an algebra (not necessarily associative).

3a. Let $\delta \in \mathrm{Der}(A)$ and $\varphi \in \mathrm{Aut}(A)$. Show that $\varphi^{-1}\delta\varphi \in \mathrm{Der}(A)$.

3b. Assume δ is a nilpotent derivation of nilpotency degree k for some natural number $k > 0$. Let $\varphi \in \mathrm{Aut}(A)$. Show that $\varphi^{-1}\delta\varphi$ is a nilpotent derivation of nilpotency degree k.

4. Let A be an algebra (not necessarily associative). Let δ be a nilpotent derivation of nilpotency degree $\leq k$ for some natural number $k > 0$. In case the characteristic p of the base field $\neq 0$, we assume that $k \leq p$. Show that $\exp(\delta)$ is an algebra automorphism of A.

5. Let L be a Lie algebra over a field of characteristic 0. For $x \in L$, recall that $\mathrm{ad}x \in \mathrm{Der}(L)$, so that we can apply the results above in case $\mathrm{ad}\, x$ is nilpotent, i.e. when x is **ad-nilpotent**. Let $\mathrm{Inn}(L)$ be the subgroup of $\mathrm{Aut}(L)$ generated by the set $\{\exp(\mathrm{ad}\, x) : x \in L \text{ and } \mathrm{ad}\, x \text{ is nilpotent}\}$. Show that $\mathrm{Inn}(L)$ is a normal subgroup of $\mathrm{Aut}(L)$.

6. Let F be a field of characteristic $\neq 2$. Let $u =$ $\in \mathrm{sl}_2(F)$.

6a. Compute the derivation ad u of $\mathrm{sl}_2(F)$; what is the matrix of ad u in some suitable basis of $\mathrm{sl}_2(F)$?

6b. Show that ad u is nilpotent.

6c. Compute exp(ad u).

6d. What is the inverse of exp(ad u)?

6e. For $y \in \mathrm{sl}_2(F)$, compute exp(ad u)(y).

6f. Show that u is nilpotent.

6g. Find exp u and $(\exp u)^{-1}$.

6h. For $y \in \mathrm{sl}_2(F)$, compute $(\exp u)y(\exp u)^{-1}$.

6i. Compare the results of 6e and 6g. They should be the same. This is not avccidental, see next exercise.

6j. Can you compute exp ad u when char(F) = 2?

7. Let L be a sub Lie algebra of gl(V). Let $x \in L$ be a nilpotent element of nilpotency class k. If characteristic p of the base field is not 0, assume that $k \leq p$.

Let λ_x and ρ_x be the right and left multiplications in the ring gl(V).

7a. Note that ad $x = \lambda_x + \rho_{-x}$ and that λ_x and ρ_{-x} are nilpotent and that they commute. Conclude that ad x is nilpotent. Thus a nilpotent element in a Lie algebra is ad-nilpotent.

7b. Show that exp ad $x = \lambda_{\exp x} \circ \rho_{\exp(-x)}$. (Hint: Use using 7a and 1d).

7c. Show that for any $y \in L$, $(\exp x)y(\exp x)^{-1} = (\exp \mathrm{ad}\, x)(y)$, in other words the action of the automorphism exp ad x on L is given by an inner conjugation.

8. Let L be a Lie algebra.

8a. Show that for $x \in L$ and $\delta \in \mathrm{Der}(L)$, $[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta x)$.

8b. Show that ad(L) is an ideal of Der(L).