Lie Algebras HW2

Ali Nesin

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1. (Lie Algebra Associated to an Associative Algebra) Let A be an associative algebra over a field F. Show that the bracket operation defined by $[x, y] = xy - yx$ turns $(A, +, [\ ,\ ])$ into a Lie algebra.

2. (**Derivations).** Let A be an algebra (not necessarily associative) over a field F. An F-linear map $\delta : A \rightarrow A$ is called a derivation, if $\delta(ab) = a\delta(b) + \delta(a)b$. Show that the set Der($A$) of derivations of A is a Lie algebra under the usual bracket operation $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$.

3. (Inner Derivations of a Lie Algebra). Let L be a Lie algebra. For $x \in L$, let ad(x) denote the map from L into L defined by $(\text{ad}(x))(y) = [x, y]$.

3a. Show that $\text{ad}(x) \in \text{Der}(L)$ for all $x \in L$.

3b. Show that $\text{Inn}(L) = \{\text{ad}(x) : x \in L\}$ is a sub Lie algebra of Der(L).

3c. Show that Inn(L) is an ideal of Der(L), i.e. that $[\text{Inn}(L), \text{Der}(L)] \leq \text{Inn}(L)$.

3d. Show that the map $\text{ad} : x \rightarrow \text{ad}(x)$ is a Lie algebra homomorphism from L into InnDer(L).

3e. What is the kernel of ad?

3f. Show that a centerless Lie algebra of finite dimension is isomorphic to a sub Lie algebra of gl(L). (In fact, by a theorem of Ado and Iwasawa, any finite dimensional Lie algebra is isomorphic to a subalgebra of gl(V) for some V, but this is harder to prove).

3g. Let $x \in \text{gl}_n(F)$ have n distinct eigenvalues $\alpha_1, ..., \alpha_n$. Show that ad(x) has n^2 eigenvalues with $\alpha_i - \alpha_j$ as the only eigenvalues.

3h. Let $x \in L$. Show that the subspace of L spanned by the eigenvectors of ad(x) is a sub Lie algebra of L.

4. (Lie Algebras of type A: Special Linear Algebras). Show that for any two $n \times n$ matrices A, B over any commutative ring, $\text{tr}(AB) = \text{tr}(BA)$. Conclude that the set $\text{sl}_n(F)$ of $n \times n$ matrices of trace 0 over a field F is a Lie subalgebra of $\text{gl}_n(F)$.

5. (Lie Algebras of type C: Symplectic Algebras). Let V be a vector space over a field F. Let $f : V \times V \rightarrow F$ be a nondegenerate bilinear skewsymmetric form on V, thus

f is bilinear
If $f(v, V) = 0$ then $v = 0$
$f(v, w) = -f(w, v)$ all $v, w \in V$

By Question #8 of HW 1, up to equivalence there is only one such f.

Let $\text{sp}(V) = \{\varphi \in \text{End}_F(V) = \text{gl}(V) : f(\varphi(v), w) = -f(v, \varphi(w))$. Show that sp($V$) is a Lie subalgebra of gl(V), in fact of sl(V) if $\dim(V) < \infty$.

6. (Lie Algebras of type B and D: Orthogonal Algebras). Let V be a vector space over a field F. Let $f : V \times V \rightarrow F$ be a nondegenerate bilinear symmetric form on V, thus

f is bilinear,
If $f(v, V) = 0$ then $v = 0$
$f(v, w) = f(w, v)$ all $v, w \in V$

Let $\text{o}(V, f) = \{\varphi \in \text{End}_F(V) = \text{gl}(V) : f(\varphi(v), w) = -f(v, \varphi(w))$. Show that o($V$) is a Lie subalgebra of gl(V), in fact of sl(V) if $\dim(V) < \infty$.