Let $R$ be a ring. A derivation $D$ is a map $D : R \to R$ such that for all $x, y \in R$
\begin{align*}
D(x + y) &= D(x) + D(y) \quad (\ast) \\
D(xy) &= D(x)y + xD(y) \quad (\ast\ast).
\end{align*}

1. Show that if $D_1$ and $D_2$ are derivations on $R$, then $D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation on $R$. (Note: This part will not be used in the sequel).

From now on we assume that the ring $R$ is commutative with identity.

2. Show that if $D$ is a derivation on a ring $R$, then
\begin{enumerate}
\item[$2a.$] $D(0) = 0.$
\item[$2b.$] $D(-x) = -D(x)$ for all $x \in R$.
\item[$2c.$] $D(n) = 0$ for all integers $n$ (we mean the image of the integer $n$ in $R$).
\item[$2d.$] $D(x^n) = nx^{n-1}D(x)$ for all $x \in R$ and $n \in \mathbb{N}$.
\item[$2e.$] $D(xyz) = D(x)yz + xD(y)z + xyD(z)$ for all $x, y, z \in R$.
\end{enumerate}

3. Show that the set of derivations is an $R$-module.

4. Let $D : R[X] \to R[X]$ be an additive map. Show that if $D$ satisfies $(\ast\ast)$ for monomials (i.e. $D(aX^nbX^m) = D(aX^n)bX^m + aX^nD(bX^m)$ for all $a, b \in R$ and $n, m \in \mathbb{N}$) then $D$ is a derivation on $R[X]$.

5. On the polynomial ring $R[X]$ define
\[(\sum \sum r_iX^i)' = \sum i r_iX^{i-1}.
\]
Show that the map $f \mapsto f'$ is a derivation on $R[X]$. (Hint: Use part 4. Note that this is the “usual” derivation).

6. Assuming $R$ is a field, what can you say about $f \in R[X]$ if $f' = 0$?

7. Let $D$ be a derivation on $R$. Extend $D$ to the polynomial ring $R[X]$ by the rule
\[D(\sum r_iX^i) = \sum D(r_i)X^i.
\]
Note that $D(X) = 0$. Show that $D$ is a derivation on $R[X]$. (Hint: Use part 4).

8. Let $D$ be a derivation on $R$. Let $u \in R[X]$ be fixed. Show that the map
\[D_u : R[X] \to R[X]
\]
given by
\[D_u(f) = D(f) + uf'
\]
for all $f \in R[X]$ is a derivation on $R[X]$ that extends the derivation $D$ of $R$ and that $D(X) = u$. (Hint: Use parts 3, 5 and 7).

9. Let $D$ be a derivation on $R$. Let $u \in R[X]$ be fixed. Show that there is a unique derivation $E$ on $R[X]$ that extends the derivation $D$ of $R$ and that $E(X) = u$. (Hint: Show that $E$, if it exists, must be as equal to $D_u$ of part 8 and use part 8 to show its existence).

10. Let $D$ be a derivation on $R$.

10a. On the polynomial ring $R[X_1, ..., X_n]$ define the derivation $D$ as in part 7.
10b. Regarding, $R[X_1, \ldots, X_n]$ as $R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n][X_i]$, as in part 5 we define a derivation that we denote by $D_i$ of by $\partial/\partial X_i$ (partial differentiation with respect to $X_i$).

10c. Let $u_1, \ldots, u_n \in R[X_1, \ldots, X_n]$. Define

$$D' = D + \sum_{i=1}^{n} u_i \partial/\partial X_i$$

Show that $D'$ is a derivation on $R[X_1, \ldots, X_n]$, that $D'|_{R} = D$ and that $D'(X_i) = u_i$ for all $i = 1, \ldots, n$.

One can show as in part 9 that the derivation $D'$ on $R[X_1, \ldots, X_n]$ defined above is the unique derivation that satisfies $D'|_{R} = D$ and $D(X_i) = u_i$ for all $i = 1, \ldots, n$.

11. (Chain Rule.) Let $S$ be a subring of $R$. Let $D$ be a derivation on $R$ such that $D|_{S} = 0$. Let $f(X_1, \ldots, X_n) \in S[X_1, \ldots, X_n]$ be a polynomial. We will show that

$$D(f(u_1, \ldots, u_n)) = \sum_{i=1}^{n} D(u_i) (\partial f/\partial X_i)(u_1, \ldots, u_n)$$

for all $u_1, \ldots, u_n$.

11c. Show that it is enough to prove this result for monomials $f$ (i.e. for polynomials of the form $f = aX_1^{r_1} \cdots X_n^{r_n}$).

11d. Show the result for all $f$.

12. (Taylor’s Expansion.) Let $f$ be a polynomial of degree $n$ over $R$ in one variable. By replacing the indeterminate of $f$ by $X + Y$, we obtain the polynomial $f(X + Y) \in R[X, Y]$. Since the degree of $f(X + Y)$ in $Y$ is still $n$, we can write,

$$f(X + Y) = f_0(X) + f_1(X)Y + \ldots + f_n(X)Y^n.$$  \hspace{1cm} (*)

We will compute the polynomials $f_i(X)$ in terms of the derivatives $f^{(j)}(X)$ for $j = 1, \ldots, n$ where $f^{(1)} = f'$ is the derivation given in part 5 and $f^{(i+1)} = (f^{(i)})$

12a. Show that

Differentiate (*) with respect to $Y$, i.e. apply $\partial/\partial Y$ to (*) $k$ times.