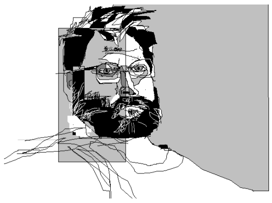
Math 212 (Algebra)



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Let R be a ring. A **derivation** D is a map $D : R \to R$ such that for all $x, y \in R$,

$$D(x + y) = D(x) + D(y) \qquad (*)$$
$$D(xy) = D(x)y + xD(y) \qquad (**).$$

1. Show that if D_1 and D_2 are derivations on R, then $D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation on R. (**Note:** This part will not be used in the sequel).

From now on we assume that the ring R is commutative with identity.

2. Show that if D is a derivation on a ring R, then

2a. $D(0) = 0$.

2b. $D(-x) = -D(x)$ for all $x \in R$.

2c. $D(n) = 0$ for all integers n (we mean the image of the integer n in R).

2d. $D(x^n) = nx^{n-1}D(x)$ for all $x \in R$ and $n \in \mathbf{N}$.

2d. $D(x^n) = nx^{n-1}D(x)$ for all $x \in R^*$ and $n \in \mathbf{Z}$.

2e. $D(xyz) = D(x)yz + xD(y)z + xyD(z)$ for all $x, y, z \in R$.

3. Show that the set of derivations is an R-module.

4. Let $D : R[X] \to R[X]$ be an additive map. Show that if D satisfies (**) for monomials (i.e. $D(aX^nbX^m) = D(aX^n)bX^m + aX^nD(bX^m)$ for all $a, b \in R$ and $n, m \in \mathbb{N}$) then D is a derivation on $R[X]$.

5. On the polynomial ring $R[X]$ define

$$(\sum_i r_iX^i)' = \sum_i ir_iX^{i-1}.$$

Show that the map $f \mapsto f'$ is a derivation on $R[X]$. (Hint: Use part 4. Note that this is the "usual" derivation).

6. Assuming R is a field, what can you say about $f \in R[X]$ if $f' = 0$?

7. Let D be a derivation on R. Extend D to the polynomial ring $R[X]$ by the rule

$$D(\sum_i r_iX^i) = \sum_i D(r_i)X^i.$$

Note that $D(X) = 0$. Show that D is a derivation on $R[X]$. (Hint: Use part 4).

8. Let D be a derivation on R. Let $u \in R[X]$ be fixed. Show that the map

$$D_u : R[X] \to R[X]$$

given by

$$D_u(f) = D(f) + uf'$$

for all $f \in R[X]$ is a derivation on $R[X]$ that extends the derivation D of R and that $D(X) = u$. (Hint: Use parts 3, 5 and 7).

9. Let D be a derivation on R. Let $u \in R[X]$ be fixed. Show that there is a unique derivation E on $R[X]$ that extends the derivation D of R and that $E(X) = u$. (Hint: Show that E, if it exists, must be as equal to D_u of part 8 and use part 8 to show its existence).

10. Let D be a derivation on R.

10a. On the polynomial ring $R[X_1, \ldots, X_n]$ define the derivation D as in part 7.

10b. Regarding, $R[X_1, \ldots, X_n]$ as $R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n][X_i]$, as in part 5 we define a derivation that we denote by D_i of by $\partial/\partial X_i$ (partial differentiation with resğpect to X_i).

10c. Let $u_1, \ldots, u_n \in R[X_1, \ldots, X_n]$. Define

$$D' = D + \sum_{i=1,\ldots,n} u_i \partial/\partial X_i$$

Show that D' is a derivation on $R[X_1, \ldots, X_n]$, that $D'|_R = D$ and that $D'(X_i) = u_i$ for all $i = 1, \ldots, n$.

One can show as in part 9 that the derivation D' on $R[X_1, \ldots, X_n]$ defined above is the unique derivation that satisfies $D'|_R = D$ and $D(X_i) = u_i$ for all $i = 1, \ldots, n$.

11. (**Chain Rule.**) Let S be a subring of R. Let D be a derivation on R such that $D|_S = 0$. Let $f(X_1, \ldots, X_n) \in S[X_1, \ldots, X_n]$ be a polynomial. We will show that

$$D(f(u_1, \ldots, u_n)) = \sum_{i=1,\ldots,n} D(u_i)\,(\partial f/\partial X_i)(u_1, \ldots u_n)$$

for all $u_1, \ldots, u_n$.

11c. Show that it is enough to prove this result for monomials f (i.e. for polynomials of the form $f = aX_1^{r_1} \cdots X_n^{r_n}$.

11d. Show the result for all f.

12. (Taylor's Expansion.) Let f be a polynomial of degree n over R in one variable. By replacing the indeterminate of f by $X + Y$, we obtain the polynomial $f(X + Y) \in R[X, Y]$. Since the degree of $f(X + Y)$ in Y is still n, we can write,

$$f(X + Y) = f_0(X) + f_1(X)Y + \ldots + f_n(X)Y^n. \qquad (*)$$

We will compute the polynomials $f_i(X)$ in terms of the derivatives $f^{(j)}(X)$ for $j = 1, \ldots, n$ where $f^{(1)} = f'$ is the derivation given in part 5 and $f^{(i+1)} = (f^{(i)})'$

12a. Show that

Differentiate (*) with respect to Y, i.e. apply $\partial/\partial Y$ to (*) k times.