Algebra (Math 212)

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Let G be a group, K a field and V a K-vector space. Recall that $\mathrm{GL}(V)$ is the set of invertible linear maps from V into itself, and that it is a group under composition.

A group homomorphism ρ from G into $\mathrm{GL}(V)$ is called a **representation** of G. We also say that V is a ***K*[*G*]-space**. For $g \in G$ and $v \in V$, instead of $\rho(g)(v)$, we may simply write gv.

1. Explain why for all $g, h \in G$, $v, w \in V$ and $\alpha \in K$, we have

$$g(v + w) = gv + gw$$
$$g(\alpha v) = \alpha gv$$
$$g(hv) = (gh)(v)$$
$$ev = v.$$

2. Conversely, suppose that a map $G \times V \to V$ that sends a pair (g, v) to an element denoted gv of V is given so that the four equalities above hold for all $g, h \in G$, $v, w \in V$ and $\alpha \in K$. Show that this map turns V into a $K[G]$-space in a natural way. (5 pts.)

2. Define the concept of the direct-sum of two $K[G]$-spaces. (5 pts.)

3. Define the concept of a $K[G]$-subspace of a $K[G]$-space. (5 pts.)

4. Show that the quotient of a $K[G]$-space by a $K[G]$-subspace is a $K[G]$-space. (5 pts.)

A $K[G]$-space V is called **irreducible** if $V \neq 0$ and if 0 and V are the only $K[G]$-subspaces of V.

5. Let V be a $K[G]$-space. Show that if G or $\dim_F(V)$ is finite then V has an irreducible $K[G]$-subspace. Assuming $|G| = n$, show that any irreducible $K[G]$-space has dimension $\leq n$. (15 pts.)

6. Assume K is algebraically closed and G is abelian. Show that any irreducible $K[G]$-space is 1-dimensional. (15 pts.)

7. Assume $|G| = n > 1$ and let V be a $K[G]$-space. Let $v \in V$ and $w = \sum_{g\in G} gv$. Show that $gw = w$ for all $g \in G$. Conclude that any irreducible $K[G]$-space has dimension $\leq n - 1$. (10 pts.)

8. Let $K = \mathbb{C}$ and $n > 0$ an integer. Find all irreducible $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$-spaces. (15 pts.)

9. Find an irreducible K[Sym(3)]-space. (10 pts.)

10. Find all irreducible $\mathbf{F}_p[\mathbb{Z}/p\mathbb{Z}]$-spaces. (20 pts.)