Group Theory Midterm
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Ali Nesin

Throughout G denotes a group.

For $X, Y \subseteq G$, let $[X, Y]$ denote the subgroup generated by all the elements of the form $[x, y] := x^{-1}y^{-1}xy$ for $x \in X$ and $y \in Y$.

1. Show that $[X, Y] = [Y, X]$.

2. Show that if H and K are subgroups of G that normalize each other then $[H, K] \leq H \cap K$.

3. Show that for $x, y, z \in G$, $[x, yz] = [x, z][x, y]^z$ and $[xy, z] = [x, z]^y[y, z]$. Conclude that if $H, K \leq G$, then H and K normalize the subgroup $[H, K]$. Conclude also that if $A \leq G$ is an abelian subgroup and if $g \in N_G(A)$, then the map $\mathrm{ad}(g) : A \to A$ defined by $\mathrm{ad}(g)(a) = [a, g]$ is a group homomorphism whose kernel is $C_A(g)$.

We define G^n and $G^{(n)}$ by induction on n:
$G^0 = G^{(0)} = G$, $G^{n+1} = [G, G^n]$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.
We let $G' = G^1 = G^{(1)}$.

4. Show that $G^{n+1} \leq G^n$ and that $G^{(n+1)} \leq G^{(n)}$. Show also that G^n and $G^{(n)}$ are characteristic subgroups of G.

5. Show that if $H \lhd G$ and G/H is abelian then $G' \leq H$. Conversely show that if $G' \leq H \leq G$, then $H \lhd G$ and G/H is abelian.

6. Let x, y, z be three elements of G. Show that
$$[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1.$$
Conclude that if H and K are two subgroups of a group G and if $[[H, K], K] = 1$, then $[H, K'] = 1$.

7. (Three Subgroup Lemma of P. Hall) Let H, K, L be three normal subgroups of G. Using # 6, show that $[[H, K], L] \leq [[K, L], H][[L, H], K]$.

8. Show that $[G^i, G^j] \leq G^{i+j+1}$ and $G^{(i)} \leq G^i$ for all i, j.

We define $Z_n(G)$ by induction on n as follows: $Z_n(G) = 1$ if $n \leq 0$ and for $n \geq 0$, $Z_{n+1}(G)$ is the unique subgroup of G that contains $Z_n(G)$ such that $Z(G/Z_n(G)) = Z_{n+1}(G)/ Z_n(G)$.

9. Show that $Z_n(G)$ is a characteristic subgroup of G for all n.

10. Show that $[G^i, Z_j] \leq Z_{j-i-1}$ and that $[Z^{i+1}, G^i] = 1$ for all i, j.

A group is said to be *solvable* if $G^{(n)} = 1$ for some n. If $G^{(n)} = 1$ but that $G^{(n-1)} \neq 1$, G is said to be solvable of class n.

A group is said to be *nilpotent* if $G^n = 1$ for some n. If $G^n = 1$ but that $G^{n-1} \neq 1$, G is said to be *nilpotent of class n*.

11. Show that a nilpotent group is solvable.

12. Let G be nilpotent of class n. Show that $G^{n-i} \leq Z_i$. Conclude that $G = Z_n$.

13. Conversely, assume that $G = Z_n$. Show that $G^i \leq Z_{n-i}$. Conclude that G is nilpotent of class n.

14. Show that G is nilpotent of class n if and only if $Z_n = G$ and $Z_{n-1} \neq G$.