Semidirect Products

Let \( U \) and \( T \) be two groups and let \( \varphi : T \to \text{Aut}(U) \), \( t \to \varphi_t \), be a group homomorphism. We will construct a new group denoted \( U \rtimes _\varphi T \), or just by \( U \rtimes T \) for short. The set on which the group operation is defined is the Cartesian product \( U \times T \), and the operation is defined as follows: \( (u, t)(u', t') = (u, \varphi_t(u'), tt') \). The reader will have no difficulty in checking that this is a group with \((1, 1)\) as the identity element. The inverse is given by the rule: \( (u, t)^{-1} = (\varphi_{t^{-1}}(u^{-1}), t^{-1}) \). Let \( G \) denote this group. \( G \) is called the semidirect product of \( U \) and \( T \) (in this order; we also omit to mention \( \varphi \)).

\( U \) can be identified with \( U \times \{1\} \) and hence can be regarded as a normal subgroup of \( G \). \( T \) can be identified with \( \{1\} \times T \) and can be regarded as a subgroup of \( G \). Then the subgroups \( U \) and \( T \) of \( G \) have the following properties: \( U \lhd G \), \( T \leq G \), \( U \cap T = 1 \) and \( G = UT \).

Conversely, whenever a group \( G \) has subgroups \( U \) and \( T \) satisfying these properties, \( G \) is isomorphic to a semidirect product \( U \rtimes _\varphi T \) where \( \varphi : T \to \text{Aut}(U) \) is given by \( \varphi_t(u) = tut^{-1} \).

Examples.

1. Let \( V \) be a vector space and \( \text{GL}(V) \) be the group of all vector space automorphisms of \( V \). The group \( V \rtimes \text{GL}(V) \) (where \( \varphi = \text{Id} \)) is a subgroup of \( \text{Sym}(V) \) as follows: \( (v, g)(w) = gw + v \).

2. The subgroup \( B_n(K) \) that consists of all the invertible \( n \times n \) upper triangular matrices over a field \( K \) is the semidirect product of \( \text{UT}_n(K) \) (upper-triangular matrices with ones on the diagonal) and \( \text{T}_n(K) \) (invertible diagonal matrices).

Exercises.

26. Let \( K \) be any field. Show that the group

\[
G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*, \ u \in K \right\}
\]

is a semidirect product of the form \( G' \rtimes T \) for some subgroup \( T \). This group is called the affine group.

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1 This is an abuse of language: every group \( G \) is split, for example as \( G = G \rtimes \{1\} \). When we use the term “split”, we have either \( U \) or \( T \) around.
27. Show that the direct product of two groups is a special case of semidirect product.

28. Let \( G = U \rtimes T \).
   a. Let \( U \leq H \leq G \). Show that \( H = U \rtimes (H \cap T) \).
   b. Let \( T \leq H \leq G \). Show that \( H = (U \cap H) \rtimes T \).
   c. Show that if \( T \) is abelian then \( G \leq U \).
   d. Show that if \( T_1 \leq T \), then \( N_U(T_1) = C_U(T_1) \).

29. Let \( G = U \rtimes T \). Let \( t \in T \) and \( x \in U \). Show that \( xt \) is \( G \)-conjugate to an element of \( T \) if and only if \( xt \) is conjugate to \( t \) if and only if \((xt)^u = t \) for some \( u \in U \) if and only if \( x \in \{ U, t^{-1} \} \).

30. Let \( G = U \rtimes T \) and let \( V \leq U \) be a \( G \)-normal subgroup of \( U \). Show that \( G/V \approx U/V \rtimes T \) in a natural way.

31. Let \( G = U \rtimes T \) and let \( V \leq U \) be a \( G \)-normal subgroup of \( U \). By Exercise 30, \( G/V = U/V \rtimes T \). Let \( t \in T \) be such that \( V = \text{ad}(t)(V) \) and \( U/V = \text{ad}(t)(U/V) \). Show that \( U = \text{ad}(t)(U) \).

32. Let \( K \) be a field and let \( n \) be a positive integer. For \( t \in K^* \) and \( x \in K \), let \( \varphi_t(x) = t^x \). Set \( G = K^* \rtimes_{\varphi} K^* \). What is the center of \( G \)? Show that \( Z_2(G) = Z(G) \). What is the condition on \( K \) that insures \( G \approx K^* \)? Show that \( G \) is isomorphic to a subgroup of \( \text{GL}_2(K) \).

41. Let \( G = \mathbb{Z}_{p^n} \rtimes \mathbb{Z}/2\mathbb{Z} \) where \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}_{p^n} \) by inversion (i.e. if \( 1 \neq i \in \mathbb{Z}/2\mathbb{Z} \) then \( \varphi(i)(g) = g^{-1} \) for all \( g \in \mathbb{Z}_{p^n} \)). Show that \( G \) is solvable of class 2, nonnilpotent but that the chain \((\mathbb{Z}_{p^n}(G))_n \in N \) is strictly increasing. Show that \( G \) is isomorphic to a Sylow 2-subgroup of \( \text{PSL}_2(K) \) where \( K \) is an algebraically closed field of characteristic \( \neq 2 \). (Recall that \( \text{SL}_2(K) \) is the group consisting of \( 2 \times 2 \) matrices of determinant 1 over \( K \), and \( \text{PSL}_2(K) \) is the factor group of \( \text{SL}_2(K) \) modulo its center that consists of the two scalar matrices \( \pm 1 \)). What is the Sylow 2-subgroup of \( \text{SL}_2(K) \) when \( \text{char}(K) = 2 \)?
on $X$ when $gx = x$ for all $x \in X$ implies $g = 1$. Note that $G/\ker(\varphi)$ acts on $X$ in a natural way: $\tilde{gx} = gx$, and this action is faithful.

Two permutation groups $(G, X)$ and $(H, Y)$ are called equivalent if there are a group isomorphism $f : G \rightarrow H$ and a bijection $\varphi : X \rightarrow Y$ such that for all $g \in G$, $x \in X$ we have $\varphi(gx) = f(g)\varphi(x)$.

Let $(G, X)$ be a permutation group. For any $Y \subseteq X$, we let $G_Y = \{g \in G : gy = y \text{ for all } y \in Y\}$. $G_Y$ is called the pointwise stabilizer of $Y$. Note that $G_Y \leq G$ is a subgroup. When $Y = \{x_1, ..., x_n\}$, we write $G_{x_1, ..., x_n}$ instead of $G_Y$. Clearly $G_Y$ is the intersection of the subgroups $G_y$ for $y \in Y$.

For $g \in G$ and $Y \subseteq X$ we define $gY = \{gy : y \in Y\}$ and the setwise stabilizer $G(Y) = \{g \in G : gY = Y\}$ of $Y$. We have $G_Y \leq G(Y)$. Finally for $A \subseteq G$, we define $F(A) = \{x \in X : ax = x \text{ for all } a \in A\}$, the set of fixed points of $A$.

**Exercise.**

45. Let $A, B \subseteq G$ and $Y, Z \subseteq X$. Then the following hold:

i. $A \subseteq G_{F(A)}$.

ii. $Y \subseteq F(G_A)$.

iii. If $A \subseteq B$ then $F(B) \subseteq F(A)$.

iv. If $Y \subseteq Z$, then $G_Z \leq G_Y$.

v. $F(G_{F(A)}) = F(A)$.

vi. $G_{F(G_Y)} = G_Y$.

We say that $G$ acts $n$-transitively on $X$ if $|X| \geq n$ and if for any pairwise distinct $x_1, ..., x_n \in X$ and any pairwise distinct $y_1, ..., y_n \in X$, there is a $g \in G$ such that $gx_i = y_i$ for all $i = 1, ..., n$. Transitive means $1$-transitive. We say that $(G, X)$ is sharply $n$-transitive if it is $n$-transitive and if the stabilizer of $n$ distinct points is reduced to $\{1\}$; in other words, if for any distinct $x_1, ..., x_n \in X$ and any distinct $y_1, ..., y_n \in X$, there is a unique $g \in G$ such that $gx_i = y_i \in X$ for all $i = 1, ..., n$. Sharply $1$-transitive actions are also called regular actions. Up to equivalence, each group has only one regular action (see Exercise 46). Clearly, for every $n$ and $|X| = n$, $(\text{Sym}(X), X)$ is sharply $n$ and also sharply $(n - 1)$-transitive. If for $g \in G$, $x \in X$, $gx = x$ implies $g = 1$, we say that the action of $G$ is free or that $G$ acts freely on $X$.

Let $X$ be a group and $G \leq \text{Aut}(X)$. Then $(G, X)$ is a permutation group. By abuse of language, one says that $G$ acts freely (resp. regularly) on $X$ if $G$ acts freely (resp. regularly) on $X'$.

Now we give the most important and, up to equivalence, the only example of transitive group actions:

**Left-Coset Representation.** Let $G$ be a group and $B \leq G$ a subgroup. Set $X = G/B$, the left-coset space. We can make $G$ act on $X$ by left multiplication: $ht(gB) = hgB$. This action is called the left-coset action, or the the left-coset representation. The kernel of this action is the core $\bigcap_{b \in G} B^b$ of $B$ in $G$, which is the maximal $G$-normal subgroup of $B$.

**Exercises**
46. Let \((G, X)\) be a transitive permutation group. Let \(x \in X\) be any point and let \(B = G_x\). Then the permutation group \((G, X)\) is equivalent to the left-coset representation \((G, G/B)\). (Hint: Let \(f = \text{Id}_G\) and \(\varphi : G/B \to X\) be defined by \(\varphi(gB) = gx\).)

47. If \(N_G(B) = B\), then the left-coset action of \(G\) on \(G/B\) is equivalent to the conjugation action of \(G\) on \(\{B^g : g \in G\}\).

48. Let \((G, X)\) be a 2-transitive group and \(B = G_x\). Then \(G = B \sqcup BgB\) for every \(g \in G \setminus B\). In particular \(B\) is a maximal subgroup of \(G\). Conversely, if \(G\) is a group with a proper subgroup \(B\) satisfying the property \(G = B \cup BgB\) for every (equivalently some) \(g \in G \setminus B\), then the permutation group \((G, G/B)\) is 2-transitive. (Hint: Assume \(G\) is 2-transitive, and let \(x \in B\) and \(B\) as in the statement. Let \(g \in G \setminus B\) be a fixed element of \(G\). Let \(h \in G \setminus B\) be any element. Since \(G\) is 2-transitive, there is an element \(b \in G\) that sends the pair of distinct points \((x, gx)\) to the pair of distinct points \((x, hx)\). Thus \(b \in B\) and \(bgx = hx\), implying \(h^{-1}bg \in B\) and \(h \in BgB\).)

49. Let \(G\) be a group and let \(H \leq G\) be a subgroup. Assume \([G:H] = n\). By considering the coset action \(G \to \text{Sym}(G/H)\) show that \([G: \bigcap_{x \in G} H^x]\) divides \(n!\). The subgroup \(\bigcap_{x \in G} H^x\) is called the core of \(H\) in \(G\).

50. Let \((G, X)\) be a permutation group. Assume \(G\) has a regular normal subgroup \(A\) (i.e. the permutation group \((A, X)\) is regular). Show that \(G = A \rtimes G_x\) for any \(x \in X\). Show that \((G, X)\) is equivalent to the permutation group \((G, A)\) where \(G = A \rtimes G_x\) acts on \(A\) as follows: For \(a \in A, h \in G_x\) and \(b \in A\), \((ah)b = ab^{h^{-1}}\). Show that \(G\) is faithful if and only if \(C_H(A) = 1\).

51. Let \((G, X)\) be a permutation group. Show that \(G_{g^{-1}x} = G_x^g\) for any \(x \in X\). Show that if \(G\) is an \(n\)-transitive group, then for any \(1 \leq i \leq n\), all the \(i\)-point stabilizers are conjugate to each other.

52. Let \((G, X)\) be a transitive permutation group. Show that if \(G\) is abelian then, for any \(x \in X\), \(G_x\) is the kernel of the action and \((G/G_x, X)\) is a regular permutation group.

53. Let \(n \geq 2\) be an integer. Show that \((G, X)\) is \(n\)-transitive if and only if \((G_x, X \setminus \{x\})\) is \((n-1)\)-transitive for any (equivalently some) \(x \in X\). State and prove a similar statement for sharply \(n\)-transitive groups.

54. Let \((G, X)\) be a permutation group. A subset \(Y \subseteq X\) is called a set of imprimitivity if for all \(g, h \in G\), either \(gY = hY\) or \(gY \cap hY = \emptyset\). If the only sets of imprimitivity are the singleton sets and \(X\), then \((G, X)\) is called a primitive permutation group. Show that a 2-transitive group is primitive. Assume that \((G, X)\) is transitive. Show that \((G, X)\) is primitive if and only if \(G_x\) is a maximal subgroup for some (equiv. all) \(x \in X\). Conclude that if \(G\) is a 2-transitive group, then \(G_x\) is a maximal subgroup. (This also follows from Exercise 48).
55. Let $G$ be a group and $B < G$ be a proper subgroup with the following properties: There is a $g \in G$ such that $G = B \cup BgB$ and if $agb = a'gb'$ for $a, a', b, b' \in B$ then $a = a'$ and $b = b'$. Show that $(G, G/B)$ is a sharply 2-transitive permutation group.

56. Let $G = A \rtimes H$ be a group where $H$ acts regularly on $A$ by conjugation (i.e. on $A^*$). Show that $G$ is a sharply 2-transitive group.

57. Let $(G, X)$ be a sharply 2-transitive permutation group, and for a fixed $x \in X$, set $B = G_x$. Show that for any fixed $g \in G \setminus B$, $G = B \sqcup BgB$ and if $agb = a'gb'$ for $a, b, a', b' \in B$, then $a = a'$ and $b = b'$. Show also that the conjugates of $B$ are disjoint from each other. Show that there are involutions that swap given any two points. Conclude that there are involutions outside of $B$.

58. Show that the group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*, u \in K \right\}$$

acts sharply 2-transitively on the set

$$X = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in K \right\}.$$

59. Show that $G = \text{PGL}_2(K) = \text{GL}_2(K)/Z$ where $Z$ is the set of scalar matrices (which is exactly the center of $\text{GL}_2(K)$) acts sharply 3-transitively on $G/B$ where $B = B_2(K)$. Show that there is a natural correspondence between $G/B$ and the set $K \cup \{\infty\}$. Transport the action of $G$ on $K \cup \{\infty\}$ and describe it algebraically.

60. Let $V$ be a vector space over a field $K$. Show that $V \rtimes \text{GL}(V)$ acts 2-transitively on $V$ (see Example 1). Show that, when $\dim_K(V) = 1$, we find the example of Exercise 58.