Semidirect Products

Let U and T be two groups and let $\varphi\colon T \to \mathrm{Aut}(U)$, $t \to \varphi_t$ be a group homomorphism. We will construct a new group denoted by $U \rtimes_\varphi T$, or just by $U \rtimes T$ for short. The set on which the group operation is defined is the Cartesian product $U \times T$, and the operation is defined as follows: $(u, t)(u', t') = (u.\varphi_t(u'), tt')$. The reader will have no difficulty in checking that this is a group with (1, 1) as the identity element. The inverse is given by the rule: $(u, t)^{-1} = (\varphi_{t^{-1}}(u^{-1}), t^{-1})$. Let G denote this group. G is called the semidirect product of U and T (in this order; we also omit to mention φ). U can be identified with $U \times \{1\}$ and hence can be regarded as a normal subgroup of G. T can be identified with $\{1\} \times T$ and can be regarded as a subgroup of G. Then the subgroups U and T of G have the following properties: $U \lhd G$, $T \leq G$, $U \cap T = 1$ and $G = UT$.

Conversely, whenever a group G has subgroups U and T satisfying these properties, G is isomorphic to a semidirect product $U \rtimes_\varphi T$ where $\varphi : T \to \mathrm{Aut}(U)$ is given by $\varphi_t(u) = tut^{-1}$.

When $G = U \rtimes T$, one says that the group G is **split**[1]; then the subgroups U and T are called each other's **complements**. We also say that T (or U) splits in G. Note that T is not the only complement of U in G: for example, any conjugate of T is still a complement of U.

When the subgroup U is abelian, it is customary to denote the group operation of U additively. In this case, it is suggestive to let $tu = \varphi_t(u)$. Then the group operation can be written as: $(u, t)(u', t') = (tu' + u, tt')$. The reader should compare this with the following formal matrix multiplication:

$$\begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t' & u' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} tt' & tu'+u \\ 0 & 1 \end{pmatrix}$$

Examples.

1. Let V be a vector space and $\mathrm{GL}(V)$ be the group of all vector space automorphisms of V. The group $V \rtimes \mathrm{GL}(V)$ (where $\varphi = \mathrm{Id}$) is a subgroup of $\mathrm{Sym}(V)$ as follows: $(v, g)(w) = gw + v$.

2. The subgroup $B_n(K)$ that consists of all the invertible $n \times n$ upper triangular matrices over a field K is the semidirect product of $\mathrm{UT}_n(K)$ (upper-triangular matrices with ones on the diagonal) and $\mathrm{T}_n(K)$ (invertible diagonal matrices).

Exercises.

26. Let K be any field. Show that the group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*,\ u \in K \right\}$$

is a semidirect product of the form $G' \rtimes T$ for some subgroup T. This group is called the **affine group**.

[1] This is an abuse of language: every group G is split, for example as $G = G \rtimes \{1\}$. When we use the term "split", we have either U or T around.

27. Show that the direct product of two groups is a special case of semidirect product.

28. Let $G = U \rtimes T$.

a. Let $U \leq H \leq G$. Show that $H = U \rtimes (H \cap T)$.

b. Let $T \leq H \leq G$. Show that $H = (U \cap H) \rtimes T$.

c. Show that if T is abelian then $G' \leq U$.

d. Show that if $T_1 \leq T$, then $N_U(T_1) = C_U(T_1)$.

29. Let $G = U \rtimes T$. Let $t \in T$ and $x \in U$. Show that xt is G-conjugate to an element of T if and only if xt is conjugate to t if and only $(xt)^u = t$ for some $u \in U$ if and only if $x \in [U, t^{-1}]$.

30. Let $G = U \rtimes T$ and let $V \leq U$ be a G-normal subgroup of U. Show that $G/V \approx U/V \rtimes T$ in a natural way.

31. Let $G = U \rtimes T$ and let $V \leq U$ be a G-normal subgroup of U. By Exercise 30, $G/V \approx U/V \rtimes T$. Let $t \in T$ be such that $V = \mathrm{ad}(t)(V)$ and $U/V = \mathrm{ad}(t)(U/V)$. Show that $U = \mathrm{ad}(t)(U)$.

32. Let K be a field and let n be a positive integer. For $t \in K^*$ and $x \in K$, let $\varphi_t(x) = t^n x$. Set $G = K^+ \rtimes_\varphi K^*$. What is the center of G? Show that $Z_2(G) = Z(G)$. What is the condition on K that insures $G' \approx K^+$? Show that G is isomorphic to a subgroup of $GL_2(K)$.

41. Let $G = \mathbf{Z}_{p^\infty} \rtimes \mathbf{Z}/2\mathbf{Z}$ where $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{Z}_{p^\infty}$ by inversion (i.e. if $1 \neq i \in \mathbf{Z}/2\mathbf{Z}$ then $\varphi_i(g) = g^{-1}$ for all $g \in \mathbf{Z}_{p^\infty}$. Show that G is solvable of class 2, nonnilpotent but that the chain $(Z\_n(G))_{n \in \mathbf{N}}$ is strictly increasing. Show that G is isomorphic to a Sylow 2-subgroup of $PSL_2(K)$ where K is an algebraically closed field of characteristic $\neq 2$. (Recall that $SL_2(K)$ is the group consisting of 2×2 matrices of determinant 1 over K, and $PSL_2(K)$ is the factor group of $SL_2(K)$ modulo its center that consists of the two scalar matrices ± 1). What is the Sylow 2-subgroup of $SL_2(K)$ when $\mathrm{char}(K) = 2$?

Permutation Groups.

Let G be a group and X a set. We say that G **acts** on X or that (G, X) is a **permutation group** if there is a map $G \times X \to X$ (denoted by $(g, x) \to g_* x$ or gx) that satisfies the following properties:

1 For all $g, h \in G$ and all $x \in X$, $g(hx) = (gh)x$.

2 For all $x \in X$, $1x = x$.

This is saying that there is a group homomorphism $\varphi: G \to \mathrm{Sym}(X)$ where $\mathrm{Sym}(X)$ is the group of all bijections of X. The kernel of φ is called the **kernel** of the action. When φ is one-to-one, the action is called **faithful**. In other words, G acts faithfully on X when $gx = x$ for all $x \in X$ implies $g = 1$. Note that $G/\ker(\varphi)$ acts on X in a natural way: $\breve{g}x = gx$, and this action is faithful.

Two permutation groups (G, X) and (H, Y) are called **equivalent** if there are a group isomorphism $f : G \to H$ and a bijection $\varphi : X \to Y$ such that for all $g \in G$, $x \in X$ we have $\varphi(gx) = f(g)\varphi(x)$.

Let (G, X) be a permutation group. For any $Y \subseteq X$, we let

$$G_Y = \{g \in G : gy = y \text{ for all } y \in Y\}.$$

G_Y is called the **pointwise stabilizer** of Y. Note that $G_Y \leq G$ is a subgroup. When $Y = \{x_1, ..., x_n\}$, we write $G_{x_1,...,x_n}$ instead of G_Y. Clearly G_Y is the intersection of the subgroups G_y for $y \in Y$.

For $g \in G$ and $Y \subseteq X$ we define $gY = \{gy : y \in Y\}$ and the **setwise stabilizer** $G(Y) = \{g \in G : gY = Y\}$ of Y. We have $G_Y \leq G(Y)$. Finally for $A \subseteq G$, we define

$$\mathrm{F}(A) = \{x \in X : ax = x \text{ for all } a \in A\},$$

the set of fixed points of A.

Exercise.

45. Let $A, B \subseteq G$ and $Y, Z \subseteq X$. Then the following hold:

i. $A \subseteq G_{F(\mathrm{A})}$.

ii. $Y \subseteq \mathrm{F}(\mathrm{G}_A)$.

iii. If $A \subseteq B$ then $\mathrm{F}(\mathrm{B}) \subseteq \mathrm{F}(A)$.

iv. If $Y \subseteq Z$, then $G_Z \leq G_Y$.

v. $\mathrm{F}(G_{\mathrm{F}(A)}) = \mathrm{F}(A)$.

vi. $G_{F(G_Y)} = \mathrm{G}_Y$.

We say that G acts ***n*-transitively** on X if $|X| \geq n$ and if for any pairwise distinct $x_1, ..., x_n \in X$ and any pairwise distinct $y_1, ..., y_n \in X$, there is a $g \in G$ such that $gx_i = y_i$ for all $i = 1, ..., n$. **Transitive** means 1-transitive. We say that (G, X) is **sharply *n*-transitive** if it is n-transitive and if the stabilizer of n distinct points is reduced to $\{1\}$; in other words, if for any distinct $x_1, ..., x_n \in X$ and any distinct $y_1, ..., y_n \in X$, there is a unique $g \in G$ such that $gx_i = y_i \in X$ for all $i = 1, ..., n$. Sharply 1-transitive actions are also called **regular actions**. Up to equivalence, each group has only one regular action (see Exercise 46). Clearly, for every n and $|X| = n$, $(\mathrm{Sym}(X), X)$ is sharply n and also sharply $(n-1)$-transitive. If for $g \in G$, $x \in X$, $gx = x$ implies $g = 1$, we say that the action of G is **free** or that G acts **freely** on X.

Let X be a group and $G \leq \mathrm{Aut}(X)$. Then (G, X) is a permutation group. By abuse of language, one says that G acts **freely** (resp. **regularly**) on X if G acts freely (resp. regularly) on X^*.

Now we give the most important and, up to equivalence, the only example of transitive group actions:

Left-Coset Representation. Let G be a group and $B \leq G$ a subgroup. Set $X = G/B$, the left-coset space. We can make G act on X by left multiplication: $h(gB) = hgB$. This action is called the **left-coset action**, or the the **left-coset representation**. The kernel of this action is the core $\bigcap_{g \in G} B^g$ of B in G, which is the maximal G-normal subgroup of B.

Exercises

46. Let (G, X) be a transitive permutation group. Let $x \in X$ be any point and let $B = G_x$. Then the permutation group (G, X) is equivalent to the left-coset representation $(G, G/B)$. (Hint: Let $f = \mathrm{Id}_G$ and $\varphi: G/B \to X$ be defined by $\varphi(gB) = gx$.)

47. If $N_G(B) = B$, then the left-coset action of G on G/B is equivalent to the conjugation action of G on $\{B^g : g \in G\}$.

48. Let (G, X) be a 2-transitive group and $B = G_x$. Then $G = B \sqcup BgB$ for every $g \in G \setminus B$. In particular B is a maximal subgroup of G. Conversely, if G is a group with a proper subgroup B satisfying the property $G = B \cup BgB$ for every (equivalently some) $g \in G \setminus B$, then the permutation group $(G, G/B)$ is 2-transitive. (Hint: Assume G is 2-transitive, and let x and B as in the statement. Let $g \in G \setminus B$ be a fixed element of G. Let $h \in G \setminus B$ be any element. Since G is 2-transitive, there is an element $b \in G$ that sends the pair of distinct points (x, gx) to the pair of distinct points (x, hx). Thus $b \in B$ and $bgx = hx$, implying $h^{-1}bg \in B$ and $h \in BgB$.)

49. Let G be a group and let $H \leq G$ be a subgroup. Assume $[G:H] = n$. By considering the coset action $G \to \mathrm{Sym}(G/H)$ show that $[G:\bigcap_{g\in G} H^g]$ divides $n!$. The subgroup $\bigcap_{g\in G} H^g$ is called the **core** of H in G.

50. Let (G, X) be a permutation group. Assume G has a regular normal subgroup A (i.e. the permutation group (A, X) is regular). Show that $G = A \rtimes G_x$ for any $x \in X$. Show that (G, X) is equivalent to the permutation group (G, A) where $G = A \rtimes G_x$ acts on A as follows: For $a \in A$, $h \in G_x$ and $b \in A$, $(ah).b = ab^{h^{-1}}$. Show that G is faithful if and only if $C_H(A) = 1$.

51 Let (G, X) be a permutation group. Show that $G_{g^{-1}x} = {G_x}^g$ for any $x \in X$. Show that if G is an n-transitive group, then for any $1 \leq i \leq n$, all the i-point stabilizers are conjugate to each other.

52. Let (G, X) be a transitive permutation group. Show that if G is abelian then, for any $x \in X$, G_x is the kernel of the action and $(G/G_x, X)$ is a regular permutation group.

53. Let $n \geq 2$ be an integer. Show that (G, X) is n-transitive if and only if $(G_x, X \setminus \{x\})$ is $(n-1)$-transitive for any (equivalently some) $x \in X$. State and prove a similar statement for sharply n-transitive groups.

54. Let (G, X) be a permutation group. A subset $Y \subseteq X$ is called a **set of imprimitivity** if for all $g, h \in G$, either $gY = hY$ or $gY \cap hY = \emptyset$. If the only sets of imprimitivity are the singleton sets and X, then (G, X) is called a **primitive** permutation group. Show that a 2-transitive group is primitive. Assume that (G, X) is transitive. Show that (G, X) is primitive if and only if G_x is a maximal subgroup for some (equiv. all) $x \in X$. Conclude that if G is a 2-transitive group, then G_x is a maximal subgroup. (This also follows from Exercise 48).

55. Let G be a group and $B < G$ be a proper subgroup with the following properties: There is a $g \in G$ such that $G = B \cup BgB$ and if $agb = a'gb'$ for a, a', b, $b' \in B$ then $a = a'$ and $b = b'$. Show that $(G, G/B)$ is a sharply 2-transitive permutation group.

56. Let $G = A \rtimes H$ be a group where H acts regularly on A by conjugation (i.e. on A^*). Show that G is a sharply 2-transitive group.

57. Let (G, X) be a sharply 2-transitive permutation group, and for a fixed $x \in X$, set $B = G_x$. Show that for any fixed $g \in G \setminus B$, $G = B \sqcup BgB$ and if $agb = a'gb'$ for a, b, a', $b' \in B$, then $a = a'$ and $b = b'$. Show also that the conjugates of B are disjoint from each other. Show that there are involutions that swap given any two points. Conclude that there are involutions outside of B.

58. Show that the group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*,\ u \in K \right\}$$

acts sharply 2-transitively on the set

$$\mathrm{X} = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in K \right\}.$$

59. Show that $G = \mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/Z$ where Z is the set of scalar matrices (which is exactly the center of $\mathrm{GL}_2(K)$) acts sharply 3-transitively on G/B where $B = \mathrm{B}_2(K)$. Show that there is a natural correspondence between G/B and the set $K \cup \{\infty\}$. Transport the action of G on $K \cup \{\infty\}$ and describe it algebraically.

60. Let V be a vector space over a field K. Show that $V \rtimes \mathrm{GL}(V)$ acts 2-transitively on V (see Example 1). Show that, when $\dim_K(\mathrm{V}) = 1$, we find the example of Exercise 58.