# Valuations

Homework
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**1.** Let $\Gamma$ be a multiplicative commutative group. An **ordering** on $\Gamma$ is a multiplicative subset $S$ of $\Gamma$ such that $\Gamma$ is the disjoint union of $S$, $S^{-1}$ and $\{1\}$. For $\alpha$, $\beta \in \Gamma$, define $\alpha < \beta$ iff $\alpha\beta^{-1} \in S$. Show that $<$ defines a total order on $\Gamma$ which is compatible with the group multiplication.

**2.** Conversely, assume that a total order which is compatible with the group operation is given on a multiplicative commutative group $\Gamma$. Show that a multiplicative subset $S$ of $\Gamma$ as above gives the same order.

**3.** Show that a group on which an ordering is given is torsion-free.

If $\Gamma$ is a group with a valuation, one attaches an element $0 \notin \Gamma$ to $\Gamma$ and extends the multiplication and the order of $\Gamma$ to $\Gamma \cup \{0\}$ as follows: $00 = 0\alpha = \alpha 0 = 0$ and $0 < \alpha$ for all $\alpha \in \Gamma$.

Let $K$ be a field. A **valuation** on $K$ is a map $|\ |$ from $K$ into $\Gamma \cup \{0\}$ where $\Gamma$ is a group with an ordering such that

i) $|x| = 0$ iff $x = 0$.
ii) $|xy| = |x||y|$ for all $x, y \in K$.
ii) $|x+y| \leq \max(|x|, |y|)$.

Replacing $\Gamma$ with $|K^*|$, we may (and will) assume that the map $|\ |$ is onto.

**4.** Show that in a field with valuation $|1| = 1$ and $|-x| = |x|$.

**5.** Show that in a field with valuation, if $|x| < |y|$ then $|x+y| = |y|$.

**6.** Let $(K, |\ |, \Gamma)$ be a field with valuation.

**6a.** Show that $o = \{x \in K : |x| \leq 1\}$ is a local ring with $\wp = \{x \in K : |x| < 1\}$ as its unique maximal ideal.

**6b.** Show that for all $x \in K^*$, either $x$ or $x^{-1}$ is in $o$.

**6c.** Show that $o^* = \{x \in K : |x| = 1\} = o \setminus \wp$.

**6d.** Show that $\Gamma \approx K^*/o^*$ canonically as groups.

**6e.** By the isomorphism above $K^*/o^*$ can be turned into a group with valuation. What is $\{s \in K^*/o^* : s < 1\}$? (This is the subgroup that corresponds to $S$).

**7.** Let $K$ be a field. A subring $o$ of $K$ is called a **valuation ring** if for any $x \in K^*$, either $x$ or $x^{-1}$ is in $o$. Let $o$ be a valuation ring of $K$.

**7a.** Show that nonunits of $o$ form an additive subgroup. (**Hint:** Let $x$, $y$ be two nonunits of $o$. We may assume that $x/y$ is in $o$ (why?). Consider the element $1 + x/y$ of $o$).

**7b.** Show that the nonunits of $o$ form an ideal $\wp$ of $o$.

**7c.** Show that $o$ is a local ring.

**7d.** Show that the image of $\wp \setminus \{0\}$ in $K^*/o^*$ is an ordering in $K^*/o^*$.

**7e.** For $x \in K^*$, let $|x|$ to be the canonical image of $x$ in $K^*/o^*$. Define $|0| = 0$ (0 is a new element not in $K^*/o^*$). Show that $(K^*, |\ |, K^*/o^*)$ defines a valuation on $K^*$.

**8.** Two valuations $|\ |_1$ and $|\ |_2$ on a field $K$ are called **equivalent** if there is an order-preserving isomorphism $\lambda : |K^*|_1 \to |K^*|_2$ such that $|x|_2 = \lambda(|x|_1)$ for all $x \in K$ (we assume that $\lambda(0) = 0$).

Show that there is a one-to-one correspondance between the valuation rings of $K$ and equivalence classes of valuations.