Algebra (Final)
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A group is a set G together with a binary operation $* : G \times G \to G$ such that

G1. For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$.

G2. There is an $e \in G$ such that for all $x \in G$, $x * e = e * x = x$.

G3. For all $x \in G$, there is a $x' \in G$ such that $x * x' = x' * x = e$.

A group is sometimes written as the pair $(G, *)$.

From now on G denotes a group. The binary operation on G will be denoted by $*$.

1. Which of the following is a group? $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}, \times)$, $(\mathbb{Q}, +)$, $(\mathbb{Q}, \times)$, $(\mathbb{Q} \setminus \{0\}, \times)$, $(\mathbb{Q}^{>0}, \times)$, $(\mathbb{Q}^{<0}, \times)$. (4 pts.)
2. Show that the set Sym(X) of bijections of a set X is a group under composition. (3 pts.)
3. Let G and H be two groups. We denote by $*$ both binary operations. Show that the binary operation defined by $(g, h) * (g', h') = (g * g', h * h')$ on $G \times H$ defines a group structure on $G \times H$. (4 pts.)
4. Let X be a set and G a group. Let Func(X, G) be the set of functions from X into G. For $\varphi, \psi \in$ Func(X, G) define $\varphi * \psi : X \to G$ by $(\varphi * \psi)(x) = \varphi(x)*\psi(x)$ for all $x \in X$. Show that the set Func(X, G) is a group under this binary operation $*$. (4 pts.)
5. Let G be a group and X a set. Let $f : G \to X$ be a bijection. For $x, y \in X$ define $x * y = f(f^{-1}(x)f^{-1}(y))$. Show that this defines a group operation on the set X. (4 pts.)
6. Show that the element e in a group G that satisfies G2 is unique. It is called the identity element of G. (2 pts.)
7. Show that if $x*y = x*z$ in a group then $y = z$. (2 pts.) Does the same hold if the hypothesis is replaced by $x*y = z*x$? (4 pts.)
8. Show that, given $x \in G$, the element $x' \in G$ that satisfies G3 is unique. (2 pts.) From now on we let $x' = x^{-1}$. Show that $e^{-1} = e$. (2 pts.) Show that $(x^{-1})^{-1} = x$. (2 pts.) We will call x^{-1} the ***inverse*** of x.
9. Find $(x * y)^{-1}$ in terms of the binary operation $*$ and the elements y^{-1} and x^{-1}. (2 pts.)
10. For $x \in G$ and $n \in \mathbb{N}$, define x^n by induction on n as follows: $x^0 = e$ and $x^{n+1} = x^n * x$. Find x^1. Show that $e^n = e$ for all $n \in \mathbb{N}$. Show that given any $x \in G$ and $n, m \in \mathbb{N}$, $x^n * x^m = x^{n+m}$. (3 pts.) Give an example of a group where $(x * y)^2$ is not always equal to $x^2 * y^2$. (2 pts.)
11. For $x \in G$ and $n \in \mathbb{N}$, define x^{-n} as $(x^n)^{-1}$. Show that $e^n = e$ for all $n \in \mathbb{Z}$. Show that given any $x \in G$ and $n, m \in \mathbb{Z}$, $x^n * x^m = x^m * x^n = x^{n+m}$. (4 pts.)
12. Show that for all $x \in G$ and $n, m \in \mathbb{Z}$, $(x^n)^m = x^{nm}$. (4 pts.)
13. For $x \in G$ if there exists a natural number $n > 0$ such that $x^n = e$, the smallest such n is called the ***order*** of x. If there is no such n, we say that the order of x is ∞. We let o(x) to denote the order of x. Show that if o(x) $\neq \infty$ and o(x) divides n then $x^n = e$. (2 pts.) Find elements of order 2, 3, 4, 5, 6 and 7 of Sym X where $X = \{1, 2, 3, 4, 5\}$ if there arae any. (5pts.)
14. Let G and H be two groups. A map $\varphi : G \to H$ is called a ***homomorphism*** (of groups) from G into H if $\varphi(x*y) = \varphi(x)*\varphi(y)$.

a) Show that the map $\varphi : G \to H$ defined by $\varphi(x) = e_H$ (the identity element of H) is always a homomorphism from G into H. (2 pts.)

b) Show that the identity map Id_G is always a homomorphism from G into G. (2 pts.)

c) Let K be another group and $\varphi : G \to H$ and $\psi : H \to K$ be two homomorphisms. Show that the composition $\psi \circ \varphi$ is a homomorphism from G into K. (2 pts.)

d) Let $\varphi : G \to H$ be a homomorphism which is also a bijection. Show that φ^{-1} is a homomorphism from H onto G. (4 pts.) Such a φ is called an ***isomorphism***.

Whenever there is an isomorphism from G into H we say that G and H are ***isomorphic*** and we write $G \approx H$. Show that $\mathrm{Func}(2, G) \approx G \times G$. (4 pts.)

e) Show that 1) $G \approx G$, 2) If $G \approx H$ then $H \approx G$, 3) If $G \approx H$ and $H \approx K$ then $G \approx K$. (4 pts.)

f) An isomorphism from G onto G is called an ***automorphism*** of G. We let $\mathrm{Aut}\, G$ be the set of automorphisms of G. Show that $\mathrm{Aut}\, G$ is a group under the composition of functions. (2 pts.)

g) Let $\varphi : G \to H$ be a homomorphism. Show that φ maps the identity element of G onto the identity element of H. (3 pts.)

h) Let $\varphi : G \to H$ be a homomorphism. Show that $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$. (3 pts.)

i) A homomorphism from a group into itself is called an ***endomorphism***. Find all endomorphisms of Sym 2, Sym 3. (4 pts.) Find all homomorphisms from Sym 3 into Sym 2. (4 pts.) Find all homomorphisms of $(\mathbb{Z}, +)$. (4 pts.)

j) Show that any two groups with n elements are isomorphic for $n = 1, 2, 3$ (4 pts.)

k) Show that there are two nonisomorphic groups G_1 and G_2 with 4 elements such that any group G with 4 elements is isomorphic to either G_1 or G_2. (8 pts.)

l) Let $a \in G$. Show that the map $\mathrm{Inn}_a : G \to G$ defined by $\mathrm{Inn}_a(x) = a^{-1}xa$ defines a group automorphism of G. (5 pts.)

m) Show that the map $\mathrm{Inn} : G \to \mathrm{Aut}\, G$ defined by $\mathrm{Inn}(a) = \mathrm{Inn}_a$ is a group homomorphism. (5 pts.)

n) Let $\varphi : G \to H$ be a homomorphism. Show that the image $\varphi(G)$ of G under φ together with the binary operation that makes H a group is a group. (5 pts.)

o) n) Let $\varphi : G \to H$ be a homomorphism. Show that the image $\{g \in G : \varphi(g) = 1\}$ together with the binary operation that makes G a group is a group. (5 pts.)

1. Prove or disprove:

1a. $\{f : \mathbb{N} \to \mathbb{N} : f$ is a bijection and $\{x \in \mathbb{N} : f(x) \neq x\}$ is finite$\}$ a subgroup of $\mathrm{Sym}(\mathbb{N})$.

1b. $\{f : \mathbb{N} \to \mathbb{N} : f$ is a bijection and $|\{x \in \mathbb{N} : f(x) \neq x\}|$ is an even integer$\}$ is a subgroup of $\mathrm{Sym}(\mathbb{N})$.

1c. $\{q \in \mathbb{Q}^{>0} : q = a/b$ and b is square-free$\}$ a subgroup of $\mathbb{Q}^*$.

2. Find the subgroup of

2a. $\mathbb{Q}^+$ generated by 2/3.

2b. $\mathbb{Q}^+$ generated by 2/3 and 4/9.

2c. $\mathbb{Q}^*$ generated by $\{1/p : p$ a prime in $\mathbb{N}\}$.

2d. $\mathbb{Q}^*$ generated by $\{1/p : p$ an odd prime in $\mathbb{N}\}$.

3. Show that the permutations $(1\ 2)$ and $(1\ 2\ \ldots\ n)$ generate $\mathrm{Sym}(n)$.

4. Write the elements of Alt(3) and of Alt(4).

5. Find $Z(\text{Sym}(4))$.

6. Show that $Z(G) \lhd G$.

7. Show that $[X, Y] = [Y, X]$ for all $X, Y \subseteq G$.

8. Let H and K be two normal subgroups of G. Show that $[H, K] \subseteq H \cap K$.

9. Let H be a nonempty finite subset of a group G closed under multiplication. Show that H is a subgroup of G.

10. Considering Sym(4) as a subgroup of Sym(5) in a natural way, find $C_{\text{Sym}(4)}((25))$.

11. Show that for all a, b, c in G,

$$[a, bc] = [a, c]\,[a, b]^c$$
$$[ab, c] = [a, c]^b[b, c].$$

15. For $a \in G$, define the function $\text{Inn}(a) : G \to G$ by

$$(\text{Inn}(a))(x) = axa^{-1}.$$

15a. Show that $\text{Inn}(a) \in \text{Aut}(G)$.
15b. Show that the map $\text{Inn} : G \to \text{Aut}(G)$ is a homomorphism of groups.
15c. What is the kernel of the homomorphism Inn?