Math 211 Final Exam
(Group Theory)
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1. Let $n > 1$ be an integer and let m be a divisor of n.
   a) Show that $\mathbb{Z}/n\mathbb{Z}$ has m elements whose order divides m. (5 pts.)
   Answer: Let $x \in \mathbb{Z}$ be such that $m\underline{x} = \underline{0}$ in $\mathbb{Z}/n\mathbb{Z}$. Then $n \mid mx$ and hence $n/m \mid x$. Therefore $x \equiv 0, n/m, 2n/m, ..., kn/m, ..., (m-1)n/m$. Thus there are exactly m elements in $\mathbb{Z}/n\mathbb{Z}$ whose order divides m.
   b) How many elements does $\mathbb{Z}/n\mathbb{Z}$ have whose order is exactly m? (5 pts.)
   Answer: By above, $\{\underline{x} \in \mathbb{Z}/n\mathbb{Z} : m\underline{x} = 0\} = \langle n/m \rangle \approx \mathbb{Z}/m\mathbb{Z}$. Thus number of elements of $\mathbb{Z}/n\mathbb{Z}$ whose order is exactly m is $\varphi(m)$.
   c) Show that a subgroup of a cyclic group is cyclic. (5 pts.)
   Answer: Let G be a cyclic group. If $G \approx \mathbb{Z}$, this must be clear at this stage of your studies. Assume $G \approx \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$. Let $H \leq \mathbb{Z}/n\mathbb{Z}$ have order m. Then $m \mid n$ and $H \leq \{h \in \mathbb{Z}/n\mathbb{Z} : mh = 0\}$. By part a, $H = \{h \in \mathbb{Z}/n\mathbb{Z} : mh = 0\}$ and H is cyclic.
2. Let p be a prime. Suppose that G has a normal and nontrivial p-subgroup. Show that G has a normal and nontrivial abelian subgroup. (5 pts.)

Answer: Let $H \lhd G$ be a p-subgroup. Since $Z(H) \neq 1$ and $Z(H)$ is characteristic in H, $Z(H) \lhd G$.

3. Let $p \leq q$ be two primes and G a group of order pq. Show that if $q \not\equiv 1 \mod p$ then G is abelian. (5 pts.)
4. Let $p < q$ be two primes and G a group of order pq^2. Show that if $q \not\equiv \pm 1 \mod p$ then G is abelian. (10 pts.)
5. Let $p < q$ be two primes with $q \equiv 1 \mod p$.
   a) Show that there are at most p nonisomorphic groups of order pq. (5 pts.)
   b) Show that the upper bound p may be attained. (5 pts.)
6. Let p and q be two primes such that $q \not\equiv 1 \mod p$. Let n be a natural number. Let G be a group of order $p^n q$.

a) Show that G has a nontrivial normal abelian subgroup. (5 pts.)
b) Show that there is a sequence $1 = G_0 \leq G_1 \leq ... \leq G_n \leq G_{n+1}$ of normal subgroups such that G_{i+1}/G_i is of prime order for each $i = 0, ..., n$. (5 pts.)

7. Give the correct mathematical definition of the following "definition": A finite group G is called ***solvable*** if either $G = 1$ or there is a nontrivial normal abelian subgroup A such that G/A is solvable. (5 pts.)
8. a) Show that if H_1 and H_2 are two subgroups of finite index of the group G, then $H_1 \cap H_2$ is a subgroup of finite index of G. (5 pts.)
   b) Show that if H is a subgroup of finite index of the group G, then H has finitely many conjugates. (5 pts.)
   c) Show that if H is a subgroup of finite index of the group G, then there is a normal subgroup N of finite index in G such that $N \subseteq H$. (5 pts.)
9. Let $H \leq G$ be a subgroup of finite index, say n. Let $X = G/H = \{xH : x \in G\}$ (the left coset space). For $g \in G$ and $xH \in X$, define $\varphi_g(xH) = gxH$.
   a) Show that φ_g is a bijection of X, so that $\varphi_g \in \mathrm{Sym}(X)$. (2 pts.)

b) For $g \in G$, let $\varphi(g) = \varphi_g \in \mathrm{Sym}(X)$. Show that φ is a group homomorphism from G into $\mathrm{Sym}(X)$. (3 pts.)
c) Show that $\mathrm{Ker}(\varphi) = \cap_{g\in G} H^g \le H$. (5 pts.)
d) Show that $[G : \mathrm{Ker}(\varphi)]$ divides $n!$ (5 pts.)
e) Compare this with #8c. (5 pts.)

10. Let G be a finite group and p, the smallest prime that divides $|G|$. Let H be a subgroup of G of index p. Show that $H \lhd G$. (20 pts.)