1. Let $n > 1$ be an integer and let $m$ be a divisor of $n$.
   a) Show that $\mathbb{Z}/n\mathbb{Z}$ has $m$ elements whose order divides $m$. (5 pts.)
   
   **Answer:** Let $x \in \mathbb{Z}$ be such that $mx = 0$ in $\mathbb{Z}/n\mathbb{Z}$. Then $n \mid mx$ and hence $n/m \mid x$. Therefore $x \equiv 0, nlm, 2n/lm, ..., knlm, ..., (m-1)nlm$. Thus there are exactly $m$ elements in $\mathbb{Z}/n\mathbb{Z}$ whose order divides $m$.
   
   b) How many elements does $\mathbb{Z}/n\mathbb{Z}$ have whose order is exactly $m$? (5 pts.)
   
   **Answer:** By above, $\{x \in \mathbb{Z}/n\mathbb{Z} : mx = 0\} = \langle nlm \rangle = \mathbb{Z}/l\mathbb{Z}$. Thus number of elements of $\mathbb{Z}/n\mathbb{Z}$ whose order is exactly $m$ is $\phi(m)$.
   
   c) Show that a subgroup of a cyclic group is cyclic. (5 pts.)

2. Let $p$ be a prime. Suppose that $G$ has a normal and nontrivial $p$-subgroup. Show that $G$ has a normal and nontrivial abelian subgroup. (5 pts.)
   
   **Answer:** Let $H \triangleleft G$ be a $p$-subgroup. Since $\mathbb{Z}(H) \neq 1$ and $\mathbb{Z}(H)$ is characteristic in $H$, $\mathbb{Z}(H) \triangleleft G$.
   
   3. Let $p \leq q$ be two primes and $G$ a group of order $pq$. Show that if $q \equiv 1 \mod p$ then $G$ is abelian. (5 pts.)
   
   4. Let $p < q$ be two primes and $G$ a group of order $pq^2$. Show that if $q \not\equiv \pm 1 \mod p$ then $G$ is abelian. (10 pts.)
   
   5. Let $p < q$ be two primes with $q \equiv 1 \mod p$.
      a) Show that there are at most $p$ nonisomorphic groups of order $pq$. (5 pts.)
      b) Show that the upper bound $p$ may be attained. (5 pts.)
   
   6. Let $p$ and $q$ be two primes such that $q \equiv 1 \mod p$. Let $n$ be a natural number. Let $G$ be a group of order $p^nq$.
      a) Show that $G$ has a nontrivial normal abelian subgroup. (5 pts.)
      b) Show that there is a sequence $1 = G_0 \leq G_1 \leq ... \leq G_n \leq G_{n+1}$ of normal subgroups such that $G_{i+1}/G_i$ is of prime order for each $i = 0, 1, ..., n$. (5 pts.)
   
   7. Give the correct mathematical definition of the following “definition”: A finite group $G$ is called **solvable** if either $G = 1$ or there is a nontrivial normal abelian subgroup $A$ such that $G/A$ is solvable. (5 pts.)
   
   8. a) Show that if $H_1$ and $H_2$ are two subgroups of finite index of the group $G$, then $H_1 \cap H_2$ is a subgroup of finite index of $G$. (5 pts.)
      b) Show that if $H$ is a subgroup of finite index of the group $G$, then $H$ has finitely many conjugates. (5 pts.)
      c) Show that if $H$ is a subgroup of finite index of the group $G$, then there is a normal subgroup $N$ of finite index in $G$ such that $N \subseteq H$. (5 pts.)
   
   9. Let $H \leq G$ be a subgroup of finite index, say $n$. Let $X = G/H = \{xH : x \in G\}$ (the left coset space). For $g \in G$ and $xH \in X$, define $\varphi_g(xH) = gxH$.
      a) Show that $\varphi_g$ is a bijection of $X$, so that $\varphi_g \in \text{Sym}(X)$. (2 pts.)
b) For $g \in G$, let $\varphi(g) = \varphi_g \in \text{Sym}(X)$. Show that $\varphi$ is a group homomorphism from $G$ into $\text{Sym}(X)$. (3 pts.)

c) Show that $\text{Ker}(\varphi) = \bigcap_{g \in G} H^g \leq H$. (5 pts.)

d) Show that $[G : \text{Ker}(\varphi)]$ divides $n!$. (5 pts.)

e) Compare this with #8c. (5 pts.)

10. Let $G$ be a finite group and $p$, the smallest prime that divides $|G|$. Let $H$ be a subgroup of $G$ of index $p$. Show that $H \triangleleft G$. (20 pts.)