Algebra Homework

1. Find all conjugacy classes in D_8.
2. Find $Z(D_8)$.
3. Find all centralizers of single elements in Sym(6).
4. Show that Sym(n) is generated by (1, 2) and (1, 2, ..., n).
5. Show that if $n > 2$, then $Z(\mathrm{Sym}(n)) = 1$.
6. Show that $Z(A \times B) = Z(A) \times Z(B)$.
7. Show that $(a, b)^{A \times B} = a^A \times b^B$.
8. Let A be an abelian group. What is the product of all its elements?
9. Show that a subgroup of index 2 is necessarily normal. Is this true for 3?
10. Let G be a group and A and B two subgroups. Show that if $G = A \cup B$ then either $G = A$ or $G = B$.
11. Let G be a group and A, B and C three proper subgroups. Assume that $G = A \cup B \cup C$. What can you say about G?
12. Let G be a group generated by a set X. Let $n \in \mathbf{N}$. Show that the set
$$\{x_1x_2 \ldots x_k : k \in \mathbf{N}, x_i \in X \text{ and } k \equiv 0 \bmod(n)\}$$
is a normal subgroup of G.
13. Let G be a group and A and B two normal subgroups of G such that $A \cap B = 1$. Show that A and B commute with each other.
14. Let G be a group and A and B two normal subgroups of G such that $A \cap B = 1$ and $G = AB$. Show that $G \approx A \times B$.
15. Let G and H be two simple groups. What are the normal subgroups of $G \times H$?
16. Find all subgroups of Sym(3) $\times$ $\mathbf{Z}/2\mathbf{Z}$.
17. Let G be a group and $H \leq G$. Define $\mathrm{N}_G(H) = \{g \in G : H^g = H\}$. Show that $H \lhd \mathrm{N}_G(H) \leq G$ and that $\mathrm{N}_G(H)$ is the largest subgroup of G containing H in which H is normal.
18. Let G be a group and $H \leq G$. Show that there is a one-to-one correspondance between $\{H^g : g \in G\}$ and $G/\mathrm{N}_G(H)$.