Algebra Homework

1. Find all conjugacy classes in $D_8$.
2. Find $Z(D_8)$.
3. Find all centralizers of single elements in Sym(6).
4. Show that Sym($n$) is generated by (1, 2) and (1, 2, ..., $n$).
5. Show that if $n > 2$, then $Z($Sym($n$)) = 1.
6. Show that $Z(A \times B) = Z(A) \times Z(B)$.
7. Show that $(a, b)^A \times B = a^A \times b^B$.
8. Let $A$ be an abelian group. What is the product of all its elements?
9. Show that a subgroup of index 2 is necessarily normal. Is this true for 3?
10. Let $G$ be a group and $A$ and $B$ two subgroups. Show that if $G = A \cup B$ then either $G = A$ or $G = B$.
11. Let $G$ be a group and $A$, $B$ and $C$ three proper subgroups. Assume that $G = A \cup B \cup C$. What can you say about $G$?
12. Let $G$ be a group generated by a set $X$. Let $n \in \mathbb{N}$. Show that the set
   \[
   \{x_1 x_2 \ldots x_k : k \in \mathbb{N}, x_i \in X \text{ and } k \equiv 0 \text{ mod}(n)\}
   \]
   is a normal subgroup of $G$.
13. Let $G$ be a group and $A$ and $B$ two normal subgroups of $G$ such that $A \cap B = 1$. Show that $A$ and $B$ commute with each other.
14. Let $G$ be a group and $A$ and $B$ two normal subgroups of $G$ such that $A \cap B = 1$ and $G = AB$. Show that $G = A \times B$.
15. Let $G$ and $H$ be two simple groups. What are the normal subgroups of $G \times H$?
16. Find all subgroups of Sym(3) $\times \mathbb{Z}/2\mathbb{Z}$.
17. Let $G$ be a group and $H \leq G$. Define $N_G(H) = \{g \in G : H^g = H\}$. Show that $H \triangleleft N_G(H) \leq G$ and that $N_G(H)$ is the largest subgroup of $G$ containing $H$ in which $H$ is normal.
18. Let $G$ be a group and $H \leq G$. Show that there is a one-to-one correspondence between $\{H^g : g \in G\}$ and $G/N_G(H)$.