Math 311

Group Theory
Midterm 1
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1. Find the number of Sylow p-subgroups of Sym(5) for all primes p. What is the cardinality of their normalizers?
2. Let n and m be two integers > 1. Find the elements of the subgroup $\{x \in \mathbf{Z}/n\mathbf{Z} : mx = 0\}$ explicitely. What is its cardinality? What is its isomorphism type?
3. Let U be a torsion abelian group. For a prime p define $U(p) = \{u \in U : u$ has order p^n for some $n\}$. Show that U is the direct sum of these subgroups.
4. Let U and V be two finite cyclic groups whose orders are relatively prime. Show that $U \times V$ is a cyclic group.
5. Let F be a field and U be a finite subgroup of F. Show that U is cyclic. (Hint : Use Questions 3 and 4 to assume that $U = U(p)$).
6. Let $\mathbf{F}_q$ be the field with q elements. (Recall that q is necesasrily a prime power, but this is not important here.) Let $\mathrm{GL}_n(q)$ be the group of invertible $n \times n$ matrices over the field $\mathbf{F}_q$. Show that $\mathrm{Z}(\mathrm{GL}_n(q))$, the center of $\mathrm{GL}_n(q)$, is the set of nonzero scalar matrices.
7. Let $\mathrm{SL}_n(q)$ be the group of $n \times n$ matrices of determinent 1 over the field $\mathbf{F}_q$. Show that $\mathrm{Z}(\mathrm{SL}_n(q))$, the center of $\mathrm{SL}_n(q)$, is the set of scalar matrices of $\mathrm{SL}_n(q)$.
8. Find the cardinalities of $\mathrm{GL}_n(q)$, $\mathrm{PGL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$ where
   $\mathrm{PGL}_n(q) = \mathrm{GL}_n(q)/\mathrm{Z}(\mathrm{GL}_n(q))$
   $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/\mathrm{Z}(\mathrm{SL}_n(q))$
9. What is the center of $\mathrm{PGL}_n(q)$?
10. Can any of the groups $\mathrm{GL}_n(q)$, $\mathrm{PGL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$ (even for different primes) be isomorphic to each other?
11. Show that $\mathrm{PSL}_2(5)$ is isomorphic to Alt(5).