Group Theory Final

Math 311
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1a. Show that the group of translations of $\mathbb{R}^n$ is a subgroup of $\mathrm{Sym}(\mathbb{R}^n)$ normalized by $\mathrm{GL}_n(\mathbb{R})$. (3 pts.)

1b. Show that the elements of $\mathrm{Sym}(\mathbb{R}^n)$ that send a line onto a line form a group that is isomorphic to the group $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ where $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ as expected. (5 pts.)

1c. Show that the above group $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ can be embedded in $\mathrm{GL}_{n+1}(\mathbb{R})$. (6 pts.)

1d. Show that any element of $\mathrm{GL}_2(\mathbb{R})$ is a product of a diagonal matrix, of an upper triangular matrix and of a rotation. (6 pts.)

1e. Let K be an algebraically closed field. Let $g \in \mathrm{GL}_2(K)$. Find all the isomorphism types of the centralizer of g in $\mathrm{GL}_2(K)$. (7 pts.)

1f. Let K be any field and let $g \in \mathrm{GL}_2(K) \setminus \mathrm{Z}(\mathrm{GL}_2(K))$. Show that the centralizer of g in $\mathrm{GL}_2(K)$ is solvable of class at most 2. Is the same true in $\mathrm{GL}_n(K)$ for $n > 2$? (7 pts.)

1g. Find all the isometries of $\mathbb{R}^2$. (10 pts.)

2. Let G be a sharply 2-transitive group acting on a set X of size n. Let $x \in X$, $T = G_x$ and $N = (G \setminus \bigcup_{g\in G} T^g) \cup \{1\}$.

2a. Show that $|G| = n^2 - n$. (3 pts.)

2b. Show that $|T| = n - 1$. (3 pts.)

2c. Show that $T^g \cap T = 1$ if $g \notin T$. (3 pts.)

2d. Show that $|N| = n$. (3 pts.)

2e. Show that $N \setminus \{1\}$ is the set of elements of G that does not fix a point of X. Conclude that N is a normal subset of G. (4 pts.)

2f. Show that if $n \in N \setminus \{1\}$, then $\mathrm{C}_G(n) \subseteq N$. (4 pts.)

2g. Show that $N \setminus \{1\}$ is one conjugacy class. (5 pts.)

2i. Find the size of $\mathrm{C}_G(n)$ for $n \in N$. (5 pts.)

2j. Conclude that N is an abelian group. (5 pts.)

2k. Show that N is an elementary abelian group. (7 pts.)

2l. Show that $G = N \rtimes T$. (2 pts.)

2m. Show that G has always an involution. (2 pts.)

2n. Show that T has an involution iff n is odd. (2 pts.)

2o. Show that N has an involution iff n is even. (2 pts.)

2p. Show that T has at most one involution, in which case this involution must be central in T. (Hint: Assume T has two involutions i and j. Let $y \neq x$ and let g carry (y, iy) onto (y, jy). Then ij^g fixes the points y and iy and g fixes x and y). (6 pts.)