Math 411

Field Theory
Midterm
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1. Let $E = \mathbb{Q}(\alpha)$ where α is the root of the equation $\alpha^3 + \alpha^2 + \alpha + 2 = 0$. Express the element $(\alpha - 1)^{-1}$ of E in the form $a\alpha^2 + b\alpha + c$ with $a, b, c \in \mathbb{Q}$.

2. Let $E = F(\alpha)$ where α is algebraic over F of odd degree. Show that $E = F(\alpha^2)$.

3. Let $F \leq E$ be a field extension. Let α and β be two elements of E which are algebraic over the field F with minimal polynomials f and g respectively. Assume that the degrees of f and g are prime to each other. Show that g is irreducible in the polynomial ring $F(\alpha)[X]$.

4. Let α be the real positive fourth root of 2. Find all subfields of $\mathbb{Q}(\alpha)$.

5. If α is a complex root of $X^6 + X^3 + 1$, find all field homomorphisms $\sigma : \mathbb{Q}(\alpha) \to \mathbf{C}$. (Hint: The polynomial is a factor of $X^9 - 1$).

6. Let E and F be two finite extensions of the field k, both contained in a larger field K. Show that $[EF : k] \leq [E : k][F : k]$. Show that the equality holds if $[E : k]$ and $[F : k]$ are prime to each other.

7. Let $f \in k[X]$ have degree n. Let K be its splitting field, i.e. $K = k[\alpha_1, ..., \alpha_n]$ where $\alpha_1, ..., \alpha_n \in K$ are such that $f = \Pi_{i=1,...,n}(X - \alpha_i)$. Show that $[K : k]$ divides n!

8. Find the splitting field of $X^{p^8} - 1$ over the field $\mathbf{F}_p$.