Group Theory

Quiz on Sylow Theorems
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Ali Nesin

G stands for a group.

1. Let H and K be two subgroups of G. For $x \in G$, the set HxK is called a **double coset** (of H and K).

1a. Show that the double cosets partition G.

1b. Assume G is finite. Show that $|G : H| = \sum_{x \in G} |K : K \cap H^x|$. (Hint: Use part a to count the number of right cosets of H in a different way).

1c. Assume G is finite and that H is a Sylow p-subgroup of G. Show that for some $x \in G$, $K \cap H^x$ is a Sylow p-subgroup of K.

From now on we always assume that G is finite.

2. Let P be a p-subgroup which is the Sylow subgroup of $N_G(P)$. Show that P is a Sylow p-subgroup of G.

3. Let P be a Sylow p-subgroup of G and let $H \triangleleft G$. Show that $H \cap P$ is a Sylow p-subgroup of H.

4 (Frattini Argument). Let $H \triangleleft G$ and P be a Sylow p-subgroup of H. Show that $G = H\, N_G(H)$.

5. Let $H \leq G$.

5a. Let P be a Sylow p-subgroup of G. Show that if $N_G(P) \leq H$, then $H = N_G(H)$.

5b. Let P be a Sylow p-subgroup of H. Show that if $N_G(P) \leq H$, then P is a Sylow p-subgroup of G.

5c. Assume that for any $1 \neq K \leq H$, $N_G(K) \leq H$. Show that $|H|$ and $|G/H|$ are prime to each other.

6. Let $|G| = p^n q$ where p and q are two distinct primes. Show that if p does not divide $q - 1$, in particular if $p \geq q$, then G is solvable[1].

7. Let $|G| = p^n q$ where p and q are two distinct primes. Show that if $q > p^n$, then G is solvable.

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[1] A famous theorem of Burnside states that a group of order $p^n q^m$ is solvable, but you cannot use this theorem in this exercise!