Math 211, Algebra MT
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I. The purpose of this first part is to prove Cauchy's Theorem, namely that every group whose order is divisible by a prime p contains an element of order p.

1. Let G be a group and $g \in G$ have order n. Let k divide n. Show that g^k has order n/k.
2. Show that a torsion group of exponent n is m-divisible for any m coprime to n.
3. Let G be a finite group. Show that for a subset X of $G \setminus Z(G)$,
$$|G| = |Z(G)| + \Sigma_{x\in X} |G/C_G(x)|.$$

Let G be a counterexample to Cauchy's Theorem of minimal order.

4. Show that no proper subgroup of G has order divisible by p.
5. Using # 3 and 4, show that G must be abelian.
6. Show that G cannot be a cyclic group.

Let $g \in G \setminus \{1\}$ and $H = \langle g \rangle$.

7. Show that H is p-divisible.
8. Show that $|G/H|$ has an element of order p.
9. Bu using #7 and 8 show that G has an element of order p, a contradiction that proves Cauchy's Theorem.

II. Show that any torsion group G (i.e. every element of G has finite order) without n-torsion elements (i.e. if $g \in G$ and $g^n = 1$ then $g = 1$) is n-divisible (i.e. for any $g \in G$ there is an $h \in G$ such that $h^n = g$).

III. Let G be a group, H a normal torsion subgroup of G and $\alpha \in G/H$ an element of some finite order n. Assume also that H has no n-torsion elements. We aim to prove that $\alpha \subseteq G$ has an element of order n.

Let $g \in \alpha$.

1. Show that $g^n \in H$.
2. Show that $g^n \in Z(C_H(g))$.
3. Show that $Z(C_H(g))$ is n-divisible.
4. Conclude that α contains an element of order n.