I. The purpose of this first part is to prove Cauchy’s Theorem, namely that every group whose order is divisible by a prime $p$ contains an element of order $p$.

1. Let $G$ be a group and $g \in G$ have order $n$. Let $k$ divide $n$. Show that $g^k$ has order $nk$.

2. Show that a torsion group of exponent $n$ is $m$-divisible for any $m$ coprime to $n$.

3. Let $G$ be a finite group. Show that for a subset $X$ of $G \setminus Z(G)$,
   \[ |G| = |Z(G)| + \sum_{x \in X} |G/C_G(x)|. \]
   Let $G$ be a counterexample to Cauchy’s Theorem of minimal order.

4. Show that no proper subgroup of $G$ has order divisible by $p$.

5. Using #3 and 4, show that $G$ must be abelian.

6. Show that $G$ cannot be a cyclic group.

Let $g \in G \setminus \{1\}$ and $H = \langle g \rangle$.

7. Show that $H$ is $p$-divisible.

8. Show that $|G/H|$ has an element of order $p$.

9. But using #7 and 8 show that $G$ has an element of order $p$, a contradiction that proves Cauchy’s Theorem.

II. Show that any torsion group $G$ (i.e. every element of $G$ has finite order) without $n$-torsion elements (i.e. if $g \in G$ and $g^n = 1$ then $g = 1$) is $n$-divisible (i.e. for any $g \in G$ there is an $h \in G$ such that $h^n = g$).

III. Let $G$ be a group, $H$ a normal torsion subgroup of $G$ and $\alpha \in G/H$ an element of some finite order $n$. Assume also that $H$ has no $n$-torsion elements. We aim to prove that $\alpha \subseteq G$ has an element of order $n$.

   Let $g \in \alpha$.

   1. Show that $g^n \in H$.

   2. Show that $g^n \in Z(C_H(g))$.

   3. Show that $Z(C_H(g))$ is $n$-divisible.

   4. Conclude that $\alpha$ contains an element of order $n$. 