Math 211 Algebra Final
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1. **[Cauchy's Theorem].** Let G be finite group and p a prime divisor of $|G|$. Let $X = \{(g_1, ..., g_p) \in G^p = G \times ... \times G : g_1, ..., g_p = 1\}$. Let $H = \langle t \rangle \approx \mathbb{Z}/p\mathbb{Z}$.
a. Show that $t(g_1, ..., g_p) = (g_p, g_1, ..., g_{p-1})$ defines an action of H on X. (5 pts.)
b. What are the possible sizes of the H-orbits? (5 pts.)
c. Show that G must have an element of order p. (10 pts.)

2. **a.** Let G be a group. Find all subgroups of $\mathbb{Z}/2\mathbb{Z} \times G$ in terms of subgroups of G. (5 pts.)
b. Let G be a simple group. Show that for any $g \in G \setminus \{1\}$, $G = \langle g^G \rangle$. (5 pts.)
c. Let $G_1, ..., G_n$ be simple nonabelian subgroups. Show that $G_1 \times ... \times G_n$ has exactly 2^n normal subgroups. (5 pts.)

d. Find the number of subgroups of $\mathbb{Z}/120\mathbb{Z}$. (5 pts.)

e. Let p be a prime. Find number of subgroups of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. (10 pts.)

3. Let $G \neq 1$ be a finite p-group where p s a prime and Φ a p-group of automorphisms of G. Show that there is a nontrivial element $g \in G$ such that $\varphi(g) = g$ for all $\varphi \in \Phi$. (15 pts.)

4. Let G be a group.
a. Show that if $A \leq G$ is an abelian subgroup and if $g \in N_G(A)$, then the map $\text{ad}(g) : A \to A$ defined by $\text{ad}(g)(a) = [a, g]$ is a group homomorphism. Find its kernel. (7 pts.)
b. Let $x, y, z \in G$. Show $[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1$. (3 pts.)
c. Conclude that if H and K are two subgroups of G and if $[[H, K], K] = 1$, then $H \leq C_G(K')$. (10 pts.)
d. **[Three Subgroup Lemma of P. Hall]** Let H, K, L be three normal subgroups of G. Using part b, show that $[[H, K], L] \leq [[K, L], H][[L, H], K]$. (5 pts.)

5. Let $H \lhd G$. Let S be a Sylow p-subgroup of H.
a. Show that there is a Sylow p-subgroup T of G such that $T \cap H = S$. (5 pts.)
b. Show that $G = HN_G(S)$. (10 pts.)

6. **a.** Show that any group of order < 60 is solvable. (25 pts.)
b. Show that a nonsolvable group of order 60 is isomorphic to Alt 5. (25 pts.)