Math 211
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G denotes always a group.

1. Show that a group of prime order is cyclic. (3 pts.)

2. Show that a subgroup of a cyclic group is cyclic. (4 pts.)

3. Show that for n and m prime to each other $\mathbb{Z}/nm\mathbb{Z} \approx \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. (2 pts.) Show the converse. (10 pts.)

4. Let the center $Z(G)$ of G be defined as $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$. It is easy to sow that $Z(G) \lhd G$. Show that if $G/Z(G)$ is a cyclic group then G is abelian. (7 pts.)

5. For $a, g \in G$ define $\text{Inn}_g(a) = gag^{-1}$. **5a.** Show that Inn_g is an automorphism of G. **5b.** Show that $\text{Inn} : G \to \text{Aut}(G)$ is a homomorphism of groups whose kernel is $Z(G)$. **5c.** Show that $\text{Inn}(G) \lhd \text{Aut}(G)$. (2 + 4 + 4 pts.)

6. Find the group structures of $\text{Aut}(\mathbb{Z}/12\mathbb{Z})$, $\text{Aut}(\text{Sym}(3))$, $\text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ and $\text{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$. (16 pts.)

7. For $X \subseteq G$ define the **centralizer** of X to be

$$C_G(X) = \{g \in G : gx = xg \text{ for all } x \in X\}.$$

It is easy to show that $C_G(X) \leq G$.

7a. Show that if $X \subseteq Y$ then $C_G(Y) \subseteq C_G(X)$. (1 pts.)

7b. Show that $X \subseteq C_G(C_G(X))$. (2 pts.)

7c. Conclude that $C_G(C_G(C_G(X))) = C_G(X)$. (5 pts.)

8. For an element $a \in G$ of a group G, define the **conjugacy class** of a to be the set

$$a^G = \{g^{-1}ag : g \in G\}$$

and the **centralizer** of a to be

$$C_G(a) = \{g \in G : ga = ag\}.$$

8a. Find an example where $C_G(a)$ is not a normal subgroup of G. (2 pts.)

8b. Show that $|G/C_G(a)| = |a^G|$. (4 pts.)

8c. Show that any two conjugacy classes are either equal or disjoint. (3 pts.)

8d. Conclude the **class formula** $|G| = |Z(G)| + \Sigma_{\text{some } g \notin Z(G)} |G|/|C_G(Z)|$. (5 pts.)

8e. Conclude that the center of a finite p-group is nontrivial. (p is a prime from now on). (4 pts.)

8f. Conclude from this and #4 that a group of order p^2 is abelian. (2 pts.)

8g. Classify all groups of order p^2. (4 pts.)

8h*. Find a nonabelian group of order p^3. (5 pts.)

8i. Classify all abelian groups of order p^3. (6 pts.)

9. Let H and K be two subgroups of G. For $x \in G$, the set HxK is called a **double coset** (of H and K).

9a. Show that the double cosets of H and K partition G. (2 pts.)

9b. Show that $|G : H| = \Sigma_{\text{some } g \in G} |K : K \cap H^x|$. (**Hint:** Use part a to count the number of right cosets of H in a different way). (8 pts.)

10a. Let A_1, ..., A_n be simple nonabelian groups. Find all normal subgroups of $A_1 \times ... \times A_n$. (5 pts.)

10b. What can you say about Aut($A \times ... \times A$) in terms of Aut(A) if A is nonabelian and simple? (15 pts.)