Group Theory Exam

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1. Let A and B be cyclic groups of order n and m respectively. Assume $(n, m) = 1$. Show that $A \oplus B$ is cyclic.

Proof: Let a and b generate A and B respectively. Then I claim that (a, b) generates $A \times B$. For this, for any $u, v \in \mathbb{Z}$, I have to find an $x \in \mathbb{Z}$ such that $(a^u, b^v) = (a, b)^x = (a^x, b^x)$, i.e. we have to solve the system

$u \equiv x \bmod n$,
$v \equiv x \bmod m$.

Since $(n, m) = 1$ there are $r, s \in \mathbb{Z}$ such that $rn + sm = 1$. Thus

$$rn(u - v) + sm(u - v) = u - v.$$

Then $x = u - rn(u - v) = v + sm(u - v)$ is a solution of the system.

2. Find number of subgroups of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (p is a prime).

Solution: $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is a vector space of dimension 2 over $\mathbb{Z}/p\mathbb{Z}$ and any subgroup is a subspace of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. There is a unique subspace of dimension 0 and a unique subspace of dimension 2. Let us count the number of subspaces of dimension 1. Each such subspace is generated by a non zero vector of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and there are $p^2 - 1$ of them. But $p - 1$ of them generate the same subspace (the ones wich are non zero multiples of each other). Hence there are $(p^2 - 1)/(p - 1) = p + 1$ such subspace. Therefore $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ has $(p + 1) + 2 = p + 3$ subgroups.

Similarly $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ has $(p^3 - 1)/(p - 1) = p^2 + p + 1$ subspaces of dimension 1. The number of a vector space of dimension 2 is equal to the number of subspaces of dimension 1 because any subspace of dimension 2 is given by the solutions (x, y, z) of an equation of the form $ax + bx + cy = 0$ for a nonzero vector $(a, b, c) \in \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Thus $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ has $2(p^2 + p + 1) + 2$ subgroups.

3. How many subgroups does $\mathbb{Z}/n\mathbb{Z}$ have?

Solution: Any subgroup of $\mathbb{Z}/n\mathbb{Z}$ is equal to $k\mathbb{Z}/n\mathbb{Z}$ for some unique k dividing n. (Why?) Thus the number of subgroups of $\mathbb{Z}/n\mathbb{Z}$ is equal to the number of divisors of n. If $n = p_1^{k_1} \ldots p_r^{k_r}$ is the prime decomposition of n then the number of divisors of n is $(k_1 + 1) \ldots (k_r + 1)$.

4. Let G be a finite p-group and Φ a p-group of automorphisms of G. Show that there is a nontrivial element $g \in G$ such that $\varphi(g) = g$ for all $\varphi \in \Phi$.

Proof: For $g \in G$, the orbit Φg is in one-to-one correspondance with $\Phi/C_\Phi(g)$ where $C_\Phi(g) = \{\varphi \in \Phi : \varphi(g) = g\}$ via the map $\varphi g \mapsto [\varphi]$ as it can be checked easily. Thus $|\Phi g|$ is a power of p. Note that $|\Phi/C_\Phi(g)| = 1$ iff $C_\Phi(g) = \Phi$ iff $\varphi(g) = g$ for all $\varphi \in \Phi$. Since G is the disjoint union of orbits and since the orbit of the identity element 1 has only 1 element, there must be other orbits with 1 element. This proves that there is a nontrivial element $g \in G$ such that $\varphi(g) = g$ for all $\varphi \in \Phi$.

5. Show that a divisible group has no proper subgroups of finite index.

Proof: Let G be a divisible group and H a subgroup of finite index. If $[G : H] = n$, we know that $\cap_{g\in G} H^g$ is a normal subgroup of index dividing n! Replacing H by this subgroup, we may assume that H is normal in G. Let n be the index. Then $g^n \in H$ for all $g \in G$. Since every $x \in G$ is an nth power, this shows that $G = H$.

6. Show that $\mathbb{Q}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are not isomorphic.

Proof: $\mathbb{Q}$ has no proper nontrivial divisible subgroup

7. Let G be an abelian group and $A \le G$. Show that if A and G/A are divisible, then G is divisible.

8. A group G is said to be ***nilpotent*** if for any proper normal subgroup $H \lhd G$, $Z(G/H) \neq 1$. Show that in a nilpotent group for any proper subgroup $H < G$, we have $H < N_G(H)$.

9. Let P be a Sylow p-subgroup of G. Show that P is characteristic in $\mathrm{N}_G(P)$ (i.e. $\varphi(P) = P$ for any automorphism φ of $N_G(P)$.) Conclude that $\mathrm{N}_G(\mathrm{N}_G(P)) = \mathrm{N}_G(P)$. Show that if G is nilpotent, $\mathrm{N}_G(P) = G$, i.e. $P \lhd G$.

10. Let $t \in G$ be an involution, i.e. an element of order 2. Let $X = \{[t, g]: g \in G\}$. (Recall that $[t, g] = t^{-1}g^{-1}tg$.

a. Show that for $x \in X$, $x^t = x^{-1}$ and that $t \notin X$. Conclude that the elements of tX are involutions.

b. Show that the map $\varphi : G/\mathrm{C}_G(t) \to X$ defined by $\varphi(g\mathrm{C}_G(t)) = [t, g^{-1}]$ is a well-defined bijection.

c. Assume from now on that G is finite and that $\mathrm{C}_G(t) = \{1, t\}$. We will show that X is an abelian subgroup without elements of order 2 and that $G = X \rtimes \{1, t\}$. Show that $|X| = |G|/2$. Show that X has no involutions. Show that $X \cap tX = \emptyset$. Show that $G = X \sqcup tX$ and that X is the set of elements of order $\neq 2$ of G. Show that X is a characteristic subset of G. Let $x \in X \setminus \{1\}$ be a fixed element. Show that t^x inverts X as well. Conclude that $1 \neq x^2 = tt^x$ centralizes X. Show that $X = \mathrm{C}_G(x^2) \le G$. Show that X is an abelian group without involutions.

11. Let G be a finite group with an involutive automorphism α (i.e. $\alpha^2 = \mathrm{Id}$) without nontrivial fixed points (i.e. $\alpha(g) = g$ implies $g = 1$). Show that G is inverted by α.

12a. Let G be a group of prime exponent p. Show that for $g \in G^*$, no two distinct elements of $\langle g \rangle$ can be conjugated in G.

b. Show that if $\exp(G) = p$, then G has at least p conjugacy classes.

c. (Reineke) Let G be a group and assume that for some $x \in G$ of finite order, we have $G = x^G \cup \{1\}$. Show that $|G| = 1$ or 2.

13. Let G be an arbitrary torsion group without involutions. Show that G is 2-divisible. Assume G has an involutive automorphism α that does not fix any nontrivial elements of G. We will show that G is abelian and is inverted by α.

a. Show that for $a, b \in G$, if $a^2 = b^2$ then $a = b$.

Let $g \in G$. Let $h \in G$ be such that $h^2 = \alpha(g)g$.

b. Show that $\alpha(h)^2 = (h^{-1})^2$. Conclude that $\alpha(h) = h^{-1}$.

c. Show that $\alpha(gh^{-1}) = gh^{-1}$. Deduce that $g = h$. This proves the result.