Group Theory Exam

January, 2007

Ali Nesin

1. Let A and B be cyclic groups of order n and m respectively. Assume $(n, m) = 1$. Show that $A \oplus B$ is cyclic.

2. Find number of subgroups of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (p is a prime).

3. How many subgroups does $\mathbb{Z}/n\mathbb{Z}$ have?

4. Let G be a finite p-group and Φ a p-group of automorphisms of G. Show that there is a nontrivial element $g \in G$ such that $\varphi(g) = g$ for all $\varphi \in \Phi$.

5. Show that a divisible group has no proper subgroups of finite index.

6. Show that $\mathbb{Q}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are not isomorphic.

7. Let G be an abelian group and $A \leq G$. Show that if A and G/A are divisible, then G is divisible.

8. A group G is said to be ***nilpotent*** if for any proper normal subgroup $H \lhd G$, $Z(G/H) \neq 1$. Show that in a nilpotent group for any proper subgroup $H < G$, we have $H < N_G(H)$.

9. Let P be a Sylow p-subgroup of G. Show that P is characteristic in $\mathrm{N}_G(P)$ (i.e. $\varphi(P) = P$ for any automorphism φ of $N_G(P)$.) Conclude that $\mathrm{N}_G(\mathrm{N}_G(P)) = \mathrm{N}_G(P)$. Show that if G is nilpotent, $\mathrm{N}_G(P) = G$, i.e. $P \lhd G$.

10. Let $t \in G$ be an involution, i.e. an element of order 2. Let $X = \{[t, g] : g \in G\}$. (Recall that $[t, g] = t^{-1}g^{-1}tg$.

a. Show that for $x \in X$, $x^t = x^{-1}$ and that $t \notin X$. Conclude that the elements of tX are involutions.

b. Show that the map $\varphi : G/\mathrm{C}_G(t) \to X$ defined by $\varphi(g\mathrm{C}_G(t)) = [t, g^{-1}]$ is a well-defined bijection.

c. Assume from now on that G is finite and that $\mathrm{C}_G(t) = \{1, t\}$. We will show that X is an abelian subgroup without elements of order 2 and that $G = X \rtimes \{1, t\}$. Show that $|X| = |G|/2$. Show that X has no involutions. Show that $X \cap tX = \emptyset$. Show that $G = X \sqcup tX$ and that X is the set of elements of order $\neq 2$ of G. Show that X is a characteristic subset of G. Let $x \in X \setminus \{1\}$ be a fixed element. Show that t^x inverts X as well. Conclude that $1 \neq x^2 = tt^x$ centralizes X. Show that $X = \mathrm{C}_G(x^2) \leq G$. Show that X is an abelian group without involutions.

11. Let G be a finite group with an involutive automorphism α (i.e. $\alpha^2 = \mathrm{Id}$) without nontrivial fixed points (i.e. $\alpha(g) = g$ implies $g = 1$). Show that G is inverted by α.

12a. Let G be a group of prime exponent p. Show that for $g \in G^*$, no two distinct elements of $\langle g \rangle$ can be conjugated in G.

b. Show that if $\exp(G) = p$, then G has at least p conjugacy classes.

c. (Reineke) Let G be a group and assume that for some $x \in G$ of finite order, we have $G = x^G \cup \{1\}$. Show that $|G| = 1$ or 2.

13. Let G be an arbitrary torsion group without involutions. Show that G is 2-divisible. Assume G has an involutive automorphism α that does not fix any nontrivial elements of G. We will show that G is abelian and is inverted by α.

a. Show that for $a, b \in G$, if $a^2 = b^2$ then $a = b$.

Let $g \in G$. Let $h \in G$ be such that $h^2 = \alpha(g)g$.

b. Show that $\alpha(h)^2 = (h^{-1})^2$. Conclude that $\alpha(h) = h^{-1}$.

c. Show that $\alpha(gh^{-1}) = gh^{-1}$. Deduce that $g = h$. This proves the result.