A regular binary tree \( \Gamma_n \) of height \( n \) is the finite graph on the set \( \{1, 2, \ldots, 2^n-1\} \) where two vertices \( a \) and \( b \) are connected if and only if \( 2^n \) divides either \((a - 2b)\) or \((b - 2a)\) and \( b \neq a \). The purpose of this exercise is to obtain some information about the automorphism group of this tree.

1. Draw \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_4 \). (To visualize better this tree, start by putting \( 2^n-1 \) to the bottom of the page and go upwards).

2. Show that the vertex \( 2^n-1 \) (called the root) is the only vertex connected to exactly two vertices, namely to \( a := 2^{n-2} \) and \( b := 2^{n-2} + 2^{n-1} \). It follows that any automorphism of \( \Gamma_n \) fixes \( 2^n-1 \) and hence either fixes or swaps \( a \) and \( b \).

3. Show that the odd numbers (called extremities) are the only vertices connected to only one vertex.

   It follows that any automorphism of \( \Gamma_n \) sends extremities to extremities.

4. Let \( H := \{ \gamma \in \text{Aut}(\Gamma_n) : \gamma(a) = a \} \) and \( H_1 := \text{Aut}(\Gamma_n) \setminus H \). Show that for any \( \alpha \in H_1 \), \( \alpha H = H_1 \).

5. Noting that \( \Gamma_n \setminus \{2^n-1\} \) is the union of two disjoint isomorphic copies of \( \Gamma_{n-1} \) (the ones above \( a \) and \( b \)), show that any \( H \simeq \text{Aut}(\Gamma_{n-1}) \times \text{Aut}(\Gamma_{n-1}) \).

6. Conclude that \( |\text{Aut}(\Gamma_n)| = 2 |\text{Aut}(\Gamma_{n-1})|^2 \).

7. Conclude that \( \text{Aut}(\Gamma_n) \) has \( 2^{2n-1} \) elements.

8. Show that \( Z(\text{Aut}(\Gamma_n)) \) consists of two elements \( \text{Id} \) and the automorphism that swaps all extremities of distance 2.
9. Show that $\Gamma_{n-1}$ is isomorphic to the subtree $\Gamma_n \setminus \{\text{odd numbers}\}$ via the map $x \mapsto x/2$. From now on we identify them.

10. Conclude that the map from $\text{Aut}(\Gamma_n)$ into $\text{Aut}(\Gamma_{n-1})$ that sends $\gamma$ to the restriction of $\gamma$ to the set of non-extremities of $\Gamma_n$ ($\simeq \Gamma_{n-1}$) is a surjection. What is its kernel?