Math 211
Algebra – Basic Group Theory
Fall 2002
Quiz I

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November 9, 2002

A **regular binary tree** Γ_n **of height** n is the finite graph on the set $\{1, 2, \ldots, 2^{n-1}\}$ where two vertices a and b are connected if and only if 2^n divides either $(a - 2b)$ or $(b - 2a)$ and $b \neq a$. The purpose of this exercise is to obtain some information about the automorphism group of this tree.

1. Draw Γ_1, Γ_2, Γ_3 and Γ_4. (To visualize better this tree, start by putting 2^{n-1} to the bottom of the page and go upwards).

2. Show that the vertex 2^{n-1} (called the **root**) is the only vertex connected to exactly two vertices, namely to $a := 2^{n-2}$ and $b := 2^{n-2} + 2^{n-1}$.

   It follows that any automorphism of Γ_n fixes 2^{n-1} and hence either fixes or swaps a and b.

3. Show that the odd numbers (called **extremities**) are the only vertices connected to only one vertex.

   It follows that any automorphism of Γ_n sends extremities to extremities.

4. Let $H := \{\gamma \in \mathrm{Aut}(\Gamma_n) : \gamma(a) = a\}$ and $H_1 := \mathrm{Aut}(\Gamma_n) \setminus H$. Show that for any $\alpha \in H_1$, $\alpha H = H_1$.

5. Noting that $\Gamma_n \setminus \{2^{n-1}\}$ is the union of two disjoint isomorphic copies of Γ_{n-1} (the ones above a and b), show that any $H \simeq \mathrm{Aut}(\Gamma_{n-1}) \times \mathrm{Aut}(\Gamma_{n-1})$.

6. Conclude that $|\mathrm{Aut}(\Gamma_n)| = 2|\mathrm{Aut}(\Gamma_{n-1})|^2$.

7. Conclude that $\mathrm{Aut}(\Gamma_n)$ has 2^{2^n-1} elements.

8. Show that $Z(\mathrm{Aut}(\Gamma_n))$ consists of two elements Id and the automorphism that swaps all extremities of distance 2.

9. Show that Γ_{n-1} is isomorphic to the subtree $\Gamma_n \setminus \{\text{odd numbers}\}$ via the map $x \mapsto x/2$. From now on we identify them.

10. Conclude that the map from $\mathrm{Aut}(\Gamma_n)$ into $\mathrm{Aut}(\Gamma_{n-1})$ that sends γ to the restriction of γ to the set of non-extremities of Γ_n ($\simeq \Gamma_{n-1}$) is a surjection. What is its kernel?