Group Theory
(Math 311)
First Midterm

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Throughout G stands for a group.

1. *Let $H, K \leq G$. Show that $\{HxK : x \in G\}$ is a partition of G.*

   Proof: The relation $x \equiv y$ defined by "$HxK = HyK$" is certainly reflexive and symmetric. Let us prove the transitivity. It is clear that $HxK = HyK$ if and only if $x \in HyK$. Thus if $x \in HyK$ and $y \in HzK$, then $x \in HHzKK \subseteq HzK$.

2. Let $H \leq G$. Show that there is a natural one to one correspondence between the left coset space of H in G and the right coset space of H in G.

   Proof: Consider the map $xH \mapsto Hx^{-1}$. This is well defined and one to one because $xH = yH$ if and only if $y^{-1}x \in H$ if and only if $y^{-1} \in Hx^{-1}$ if and only if $Hy^{-1} = Hx^{-1}$. It is also onto.

3. Let $H, K \leq G$. Show that $xH \cap yK$ is either empty or of the form $z(H \cap K)$ for some $z \in G$.

   Proof: Assume $xH \cap yK \neq \emptyset$. Let $z \in xH \cap yK$. Then $xH = zH$ and $yK = zK$. So $xH \cap yK = zH \cap zK = z(H \cap K)$.

4. a) Show that the intersection of two subgroups of finite index is finite.

   Proof: Let H and K be two subgroups of index n and m of a group G. Then for any $x \in G$, $x(H \cap K) = xH \cap xK$ and there are at most n choices for xH and m choices for xK. Hence $[G : H \cap K] \leq nm$.

   b) If $[G : H] = n$ and $[G : K] = m$, what can you say about $[G : H \cap K]$?

   Proof: If $C \leq B \leq A$ and if the indices are finite then $[A : C] = [A : B][B : C]$ because cosets of C partition B and cosets of B partition A, i.e. if $B = \sqcup_{i=1}^{r} b_i C$ and $A = \sqcup_{j=1}^{s} a_j B$, then $A = \sqcup_{i=1}^{r} \sqcup_{j=1}^{s} b_i a_j C$.

Thus $[G : K \cap H] = [G : H][H : H \cap K] = [G : K][K : H \cap K]$. It follows that n and m both divide $[G : K \cap H]$, hence $\mathrm{lcm}(n, m)$ divides $[G : K \cap H]$. Further in part (a) we have seen that $[G : K \cap H] \leq mn$.

5. Let G be a group and $H \leq G$ a subgroup of index n. Let $X = G/H$ be the left coset space. For $g \in G$, define $\tilde{g} : G/H \longrightarrow G/H$ by $\tilde{g}(xH) = gxH$ for $x \in G$.

   a) Show that $\tilde{g} \in \mathrm{Sym}(X)$.

   Proof: Nothing can be clearer.

   b) Show that $\tilde{} : G \longrightarrow \mathrm{Sym}(X)$ is a homomorphism of groups.

   Proof: Nothing can be clearer.

   c) Show that $\mathrm{Ker}(\tilde{})$ is the largest normal subgroup of G contained in H.

   Proof: $\mathrm{Ker}(\tilde{})$ is certainly a normal subgroup of G. Also $\mathrm{Ker}(\tilde{}) = \{g \in G : \tilde{g} = \mathrm{Id}\} = \{g \in G : gxH = xH \text{ for all } x \in G\} = \{g \in G : x^{-1}gx \in H \text{ for all } x \in G\} = \{g \in G : g \in xHx^{-1} \text{ for all } x \in G\} = \cap_{x \in G} H^x$. It is now clear that $\mathrm{Ker}(\tilde{})$ is the largest normal subgroup of G contained in H.

   d) Show that $[G : \mathrm{Ker}(\tilde{})]$ divides $n!$.

   Proof: By above $G/\mathrm{Ker}(\tilde{})$ embeds in $\mathrm{Sym}(G/H) \simeq \mathrm{Sym}(n)$.

   e) Conclude that there is an $m \in \mathbb{N} \setminus \{0\}$ such that for every $g \in G$, $g^m \in H$.

   Proof: Take $m = n!$.

   f) Conclude that a divisible group cannot have a proper subgroup of finite index.

   Proof: Let G be a divisible group and $H \leq G$ a subgroup of index n. Let $g \in G$. Let $h \in G$ be such that $g = h^{n!}$. By the above, $g = h^{n!} \in H$. So $G = H$.

6. Let $a \in G$. Show that there is a one to one correspondence between the left coset space $G/C_G(a)$ and the conjugacy class a^G.

   Proof: By question 2, we may assume that $G/C_G(a)$ stands for the right coset space $\{C_G(a)g : g \in G\}$. It is easy to check that the map $C_G(a)g \mapsto a^g$ is a well-defined bijection between $G/C_G(a)$ and a^G.

7. a) Let X be any set. Show that in $\mathrm{Sym}(X)$ two elements are conjugate if and only if they have the same cycle structures.

   Proof: Suppose α and β have the same cycle structures. Write α and β as the product of disjoint cycles one under another in such a way that the cycles of the same length are one on top of another:

$$\alpha = (\ldots a_1\, a_2\, a_3 \ldots) \ldots$$

$$\beta = (\ldots b_1\, b_2\, b_3 \ldots) \ldots$$

Now let $g \in \mathrm{Sym}(X)$ send a's to b's in that order. Now $g\alpha g^{-1}(b_i) = g\alpha(a_i) = g(a_{i+1}) = b_{i+1}$, hence $g\alpha g^{-1} = \beta$.

Conversely, suppose that $g\alpha g^{-1} = \beta$. Suppose for example that $(a_1 \ \ldots, a_n)$ is a cycle of α. It follows easily that $(g(a_1) \ \ldots, g(a_n))$ is a cycle of β.

b) Show that the elements $(01)(23)(45)\ldots$ and $(12)(34)(56)\ldots$ of $\mathrm{Sym}(\omega)$ are not conjugate.

Proof: They do not have the same cycle structure. The first one has no cycles of length 1, the second one has one cycle of length 1.

c) Can $(01)(23)(45)\ldots$ and $(12)(34)(56)\ldots$ of $\mathrm{Sym}(\omega)$ be conjugate in a larger group?

Answer: Yes! In $\mathrm{Sym}(\mathbb{Z})$... Because in $\mathrm{Sym}(\mathbb{Z})$ they have the same cycle.

8. Compute $|C_{\mathrm{Sym}(n)}(g)|$ for $g \in \mathrm{Sym}(n)$ and $n = 2, 3, 4, 5, 6$.

   Answer: We compute the sizes of conjugacy classes. This is enough by part 6.

n	2	3	4	5	6
Id_n	1	1	1	1	1
(12)	1	3	6	10	15
(123)		2	8	20	40
(12)(34)			3	15	45
(1234)			6	30	90
(12)(345)				20	120
(12345)				24	144
(12)(34)(56)					15
(12)(3456)					90
(123)(456)					40
(123456)					120
Total	2	6	24	120	720

9. Show that $C_{\mathrm{Sym}(n)}(12\ldots n)$ is cyclic of order n.

   Proof: Clearly $\langle(12\ldots n)\rangle \le C_{\mathrm{Sym}(n)}(12\ldots n)$. By parts 6 and 7, $|C_{\mathrm{Sym}(n)}(12\ldots n)| = n!/(12\ldots n)^G| = n!/(n-1)! = n$. It follows that $\langle(12\ldots n)\rangle = C_{\mathrm{Sym}(n)}(12\ldots n)$.

10. Let X be a set and $g \in \mathrm{Sym}(X)$. let $Y = \{x \in X : g(x) \neq x\}$. Show that $C_{\mathrm{Sym}(X)}(g) \simeq C_{\mathrm{Sym}(Y)}(g) \times \mathrm{Sym}(X \setminus Y)$.

    Proof: We view $\mathrm{Sym}(Y)$ and $\mathrm{Sym}(X \setminus Y)$ as subgroups of $\mathrm{Sym}(X)$ in the obvious way.

    We certainly have $C_{\mathrm{Sym}(Y)}(g), \mathrm{Sym}(X \setminus Y) \le C_{\mathrm{Sym}(X)}(g)$. Also $C_{\mathrm{Sym}(Y)}(g) \cap \mathrm{Sym}(X \setminus Y) = 1$ and the elements of $C_{\mathrm{Sym}(Y)}(g)$ commute with the elements of $\mathrm{Sym}(X \setminus Y)$. Thus $C_{\mathrm{Sym}(Y)}(g) \times \mathrm{Sym}(X \setminus Y) = \langle C_{\mathrm{Sym}(Y)}(g), \mathrm{Sym}(X \setminus Y) \le C_{\mathrm{Sym}(X)}(g)$.

    Conversely, let $c \in C_{\mathrm{Sym}(X)}(g)$. Then $gc = cg$. If $x \in X \setminus Y$, we get $g(x) = gc(x) = cg(x)$, so that c fixes $g(x)$ and hence g sends $X \setminus Y$ into

$X \setminus Y$. Similarly, if $x \in Y$, then $g(x) \neq gc(x) = cg(x)$, so that $g(x) \in Y$ and hence g sends Y into Y. Now we can write $g = ab$ where $a \in \mathrm{Sym}(X \setminus Y)$ and $b \in \mathrm{Sym}(Y)$. Now $b = a^{-1}g \in \mathrm{Sym}(X \setminus Y) C_{\mathrm{Sym}(X)}(g) \leq C_{\mathrm{Sym}(X)}(g)$. It follows that $b \in C_{\mathrm{Sym}(Y)}(g)$. $\square$

11. Let $g = (01)(234)(5678)(9\,10\,11\,12\,13)\ldots$. What is the group structure of $C_{\mathrm{Sym}(\omega)}(g)$?

    Answer: By parts 9 and 10, $C_{\mathrm{Sym}(\omega)}(g) \simeq \oplus_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$.

12. Let $a = (123)(456)(789)(10\,11\,12)$ Show that $C_{\mathrm{Sym}(12)}(a) \simeq (\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathrm{Sym}(4)$.

    Proof: We embed $\mathrm{Sym}(4)$ in $\mathrm{Sym}(12)$ via

$$\begin{array}{lll}
\mathrm{Id}_3 & \mapsto & \mathrm{Id}_{12} \\
(12) & \mapsto & (14)(25)(36) \\
(13) & \mapsto & (17)(28)(39) \\
(23) & \mapsto & (47)(58)(79) \\
(123) & \mapsto & (147)(258)(369) \\
\text{etc} & &
\end{array}$$

    In other words, we view $\mathrm{Sym}(4)$ as the permutations of the four cycles (123), (456), (789), $(10\,11\,12)$.

    It is clear that the image of $\mathrm{Sym}(4)$ in $\mathrm{Sym}(12)$ is in $C_{\mathrm{Sym}(12)}(a)$.

    Let $g \in C_{\mathrm{Sym}(12)}(a)$. Then g permutes the four cycles (123), (456), (789), $(10\,11\,12)$. Hence there is an $h \in \mathrm{Sym}(4)$ (or in its image) such that $h^{-1}g$ preserves the four cycles. Hence $h^{-1}g$ is in the centralizer of these four cycles, which is equal to $\langle (123), (456), (789), (10\,11\,12) \rangle$ and to

$$\langle (123) \rangle \oplus \langle (456) \rangle \oplus \langle (789) \rangle \oplus \langle (10\,11\,12) \rangle,$$

    hence isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.

    Thus $C_{\mathrm{Sym}(12)}(a) = (C_{\mathrm{Sym}(12)}((123), (456), (789), (10\,11\,12)))\,\mathrm{Sym}(4) \simeq (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \mathrm{Sym}(4)$. $\square$

13. Show that, except for $n = 4$, the centralizer of a transposition is the smallest centralizer of involutions ($\equiv$ elements of order 2) in $\mathrm{Sym}(n)$.

    Proof: An involution is a product of disjoint transpositions. For $2 \leq 2i \leq n$, let $a_i = (12)(34)\ldots(2i-1, 2i)$. We want to show that $|C_{\mathrm{Sym}(n)}(a_i)| \geq |C_{\mathrm{Sym}(n)}(a_1)|$ for all i and all $n \neq 4$. By part 8, we may assume that $n \geq 5$. By part 6, it is enough to show that $|a_i^{\mathrm{Sym}(n)}| \geq |a_1^{\mathrm{Sym}(n)}|$ for all i and all $n \geq 5$. By part 7,

$$\begin{array}{lll}
|a_i^{\mathrm{Sym}(n)}| & = & \begin{pmatrix} n \\ 2 \end{pmatrix} \begin{pmatrix} n-2 \\ 2 \end{pmatrix} \cdots \begin{pmatrix} n-2i+2 \\ 2 \end{pmatrix} / i! \\
 & = & \frac{n!}{2^i (n-2i)! i!}.
\end{array}$$

Hence we have to show that

$$\frac{n!}{2^i(n-2i)!i!} \geq \frac{n!}{2(n-2)!},$$

i.e. that

$$(n-2)! \geq 2^{i-1}(n-2i)!i!$$

for all $n \geq 5$ and all i such that $2 \leq 2i \leq n$. We proceed by induction on n. We know that the inequality must hold for $n = 5$ (by part 8). Assume for n. We have to show that

$$(n-1)! \geq 2^{i-1}(n+1-2i)!i!$$

for all i such that $2 \leq 2i \leq n+1$. Then for all i such that $2 \leq 2i \leq n$, we have

$$\begin{array}{rcl} (n-1)! & = & (n-1)(n-2)! \qquad\qquad\qquad \geq \quad (n-1)2^{i-1}(n-2i)!i! \\ & \geq & (n+1-2i)2^{i-1}(n-2i)!i! \quad \geq \quad 2^{i-1}(n+1-2i)!i! \end{array}$$

It remains to prove the case $2i = n+1$, or $n = 2i-1$, i.e. we have to show that $(2i-2)! \geq 2^{i-1}i!$ for $i \geq 3$. This is easy to show.

14. Find and prove a similar statement for $\mathrm{Alt}(n)$.

    Proof: We will do this in class.