Group Theory

Summer School, Exam 2

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I. General.

1. Let $g \in G$ have order n. Let $d \mid n$ and $q = n/d$. Show that g^q has order d and g^d has order q.

2. Let A be an abelian group. Let $A[r] = \{a \in A : a^r = 1\}$. Show that $A[r] \le A$.

3. Let A be an abelian group. Let $A^r = \{a^r : a \in A\}$. Show that $A^r \le A$.

4. Let A be an abelian group of exponent[1] nm where $(n, m) = 1$. Show that

$A^n = A[m]$,

$A^m = A[n]$

$A = A[n] \oplus A[m]$

5. Let A and B be cyclic groups of order n and m respectively. Assume $(n, m) = 1$. Show that $A \oplus B$ is cyclic.

6. Find all subgroups of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (p is a prime).

7. How many subgroups does $\mathbb{Z}/n\mathbb{Z}$ have?

8. Find a finite group G and an endomorphism φ of G such that $\varphi(Z(G)) \neq Z(G)$.

9. Let G be a finite p-group and Φ a p-group of automorphisms of G. Show that there is a nontrivial element $g \in G$ such that $\varphi(g) = g$ for all $\varphi \in \Phi$.

10. Are there simple nonabelian groups of order ≥ 61 and < 120?

II. Divisible Groups.

1. Show that a quotient of a divisible[2] group by a normal subgroup is divisible.

2. Show that a divisible group has no proper subgroups of finite index.

3. Show that $A \oplus B$ is divisible iff A and B are divisible.

4. Show that the group $\mathbb{Q}$ has no proper, nontrivial divisible subgroups. Conclude that $\mathbb{Q}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are not isomorphic. Generalize this to $\oplus_{i=1,\dots,n} \mathbb{Q}$.

5. Let G be an abelian group and $A \le G$. Show that if A and G/A are divisible, then G is divisible.

6*. Show that the statement above is false if G is nonabelian.

7. Show that an abelian group has a unique maximal divisible subgroup.

7*. Let G be an abelian group and D a divisible subgroup of G. Show that every subgroup K of G disjoint from D can be extended to a complement of D. (**Hint:** Using Zorn's Lemma, find a subgroup H containing K, disjoint from D and maximal for these properties. The maximality of H insures that $G = D \oplus H$.)

8. Show that a (not necessarily abelian) torsion group that has no elements of order p where p is a prime is p-divisible[3]. Show that a group which is p-divisible for all primes p is divisible.

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[1] A group G has exponent n if $g^n = 1$ for all $g \in G$ and if n is the least such integer > 0.

[2] A group G is said to be divisible if for all $g \in G$ and $n > 0$ there is an $h \in G$ such that $h^n = g$.

[3] A group is said to be p-divisible if for all $g \in G$ there is an $h \in G$ such that $h^p = g$.