# Algebra I (Math 221)
# Final

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Let $G$ be a group.

## 1 Characteristic Subgroups

A subgroup $H$ of $G$ is called **characteristic** if for any automorphism $\phi$ of $G$, $\phi(H) \leq H$; in this case, one writes $H \operatorname{char} G$.

1. Show that if $H \operatorname{char} G$, then $\phi(H) = H$ for all $\phi \in \operatorname{Aut}(G)$.
2. Show that if $H \operatorname{char} G$, then $\operatorname{Aut}(G)_H := \{\phi \in \operatorname{Aut}(G) : \phi_{|H} = Id_H\} \lhd \operatorname{Aut}(G)$.
3. Show that if $H \operatorname{char} G$ then $H \lhd G$.
4. Show that if $K \operatorname{char} H \lhd G$ then $K \lhd G$.
5. Let $X \subseteq G$, Suppose that $\phi(X) \subseteq X$ for all $\phi \in \operatorname{Aut}(G)$. Show that $\langle X \rangle \operatorname{char} G$.
6. Show that for any subset $X$ of $G$ there is a unique smallest characteristic subgroup of $G$ that contains $X$.
7. Let $p$ be a prime. Can a group of order $p^3$ have a characteristic subgroup of order $p^2$?
8. Let $H \operatorname{char} G$. Show that there is a canonical group homomorphism from $\operatorname{Aut}(G)$ into $\operatorname{Aut}(G/H)$.

## 2 Derived Subgroups

For $x, y \in G$, let $[x, y] = x^{-1}y^{-1}xy$. For $X, Y \subseteq G$, let $[X, Y] : \langle [x, y] : x \in X, y \in Y \langle$.

For $n \in \mathbb{N}$, define $G^n$ and $G^{(n)}$ by induction on $n$ as follows:

$$
\begin{aligned}
&G^0 = G^{(0)} = G\\
&G' = G^1 = G^{(1)} = [G, G]\\
&G^{n+1} = [G, G^n]\\
&G^{(n+1)} = [G^{(n)}, G^{(n)}] = (G^{(n)})'
\end{aligned}
$$

1. Let $X$, $Y \subseteq G$. Show that $[X, Y] = [Y, X]$.
2. Let $X$, $Y \subseteq G$. Show that if $\phi(X) \subseteq X$ and $\phi(Y) \subseteq Y$ for all $\phi \in \mathrm{Aut}(G)$, then $[X, Y] \operatorname{char} G$.
3. Show that $G^{(n+1)} \leq G^{(n)}$.
4. Show that for all $n \in \mathbb{N}$, $G^n \operatorname{char} G$ and $G^{(n)} \operatorname{char} G$.
5. Show that $G^{n+1} \leq G^n$.
6. Show that $G^n/G^{n+1} \leq Z(G/G^{n+1})$.
7. Show that $G/G'$ is an abelian group.
8. Show that if $H \lhd G$ is such that $G/H$ is abelian, then $G' \leq H$. Conclude that $G'$ is the smallest normal subgroup of $G$ such that $G/G'$ is abelian.
9. Suppose $G' \leq H \leq G$. Show that $H \lhd G$ and that $G/H$ is abelian.
10. Let $H \lhd G$. Show (carefully, in full details) that $(G/H)^n = G^n H/H$ and $(G/H)^{(n)} = G^{(n)} H/H$.