Group Theory

Summer School, Exam 2
Gümüşlük, August 3, 2001
Ali Nesin

1. Let G be a set together with an associative binary operation $(x, y) \mapsto xy$.

1a. Assume that for all $a, b \in G$ there are unique $x, y \in G$ such that $ax = ya = b$. Show that G is a group under this binary operation.

1b. Assume that for all $a, b \in G$ there is a unique $x \in G$ such that $ax = b$. Is G necessarily a group under this binary operation?

2. Let H and K be two subgroups of G. Show that for x and y in G, $xH \cap yK$ either is empty or a coset of $H \cap K$.

3. Let H and K be two subgroups of G. An H–K-coset of G is a subset of G of the form HxK for some $x \in G$. Show that the H–K-cosets of G partition G.

4. Let A and B be two simple nonabelian groups[1]. Find all normal subgroups of $A \times B$.

5. Find a group G that has a proper subgroup isomorphic to itself.

6. Let G be a group generated by $x_1, \ldots, x_n$ and let $m \in \mathbb{N}$. Show that the set

$$G[m] = \left\{ x_{i_1}^{r_1} \ldots x_{i_k}^{r_k} : k \in \mathrm{N}, \sum_{j=1}^{k} r_j \equiv 0 \,(\mathrm{mod}\, m) \right\}$$

is a normal subgroup of index at most m of G. Show that $G/G[m]$ is a cyclic group whose order divides m.

7. Classify all abelian groups which satisfy DCC on subgroups[2].

8a. Let A and B be two groups. Assume B is abelian. Let $\mathrm{Hom}(A, B)$ be the set of group homomorphisms from A into B. For $f, g \in \mathrm{Hom}(A, B)$ define $f \cdot g$ by the rule

$$(f \cdot g)(a) = f(a)g(a) \qquad (a \in A).$$

Show that the set of group endomorphisms $\mathrm{Hom}(A, B)$ from A into B form an abelian group under the addition of functions.

8b. Let n and m be two integers > 0. Show that $\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \approx m'\mathbf{Z}/m\mathbf{Z} \approx \mathbf{Z}/d\mathbf{Z}$ where $m' = m/d$ and $d = \gcd(m, n)$.

9. Find $C_{\mathrm{Sym}(n)}(1\ 2)$.

10. Let H and K be two subgroups of finite index of G. Show that $H \cap K$ has also finite index in G. Show that H has finitely many conjugates in G. Conclude that if a group has a subgroup of finite index, then it has a normal subgroup of finite index.

---

[1] A group G is called simple if 1 and G are the only normal subgroups of G.

[2] A group is said to satisfy descending chain condition (DCC) on subgroups, if there is no infinite descending chain of distinct subgroups.

11. Show that $\mathbb{Q}^+$ and $\mathbb{Q}^*$ are not finitely generated.

12. Let P be a Sylow p-subgroup of G. Show that P is characteristic in $N_G(P)$. Conclude that $N_G(N_G(P)) = N_G(P)$. Show that if G is nilpotent, $N_G(P) = G$, i.e. $P \lhd G$.

13. Let $t \in G$ be an involution[3]. Let $X = \{[t, g]: g \in G\}$

a. Show that for $x \in X$, $x^t = x^{-1}$ and that $t \notin X$. Conclude that the elements of tX are involutions.

b. Show that the map $\varphi : G/C_G(t) \to X$ defined by $\varphi(gC_G(t)) = [t, g^{-1}]$ is a well-defined bijection.

c. Assume from now on that G is finite and that $C_G(t) = \{1, t\}$. We will show that X is an abelian 2'-subgroup[4] and $G = X \rtimes \{1, t\}$. By part b, $|X| = |G|/2$. By part a and by assumption, X has no involutions. Therefore $X \cap tX = \varnothing$. Conclude that $G = X \sqcup tX$ and that X is the set of elements of order $\neq 2$ of G. Therefore, X is a characteristic subset of G. Let $x \in X \setminus \{1\}$ be a fixed element. Conclude that t^x inverts X as well (replace t by t^x). Conclude that $1 \neq x^2 = tt^x$ centralizes X. Therefore $X = C_G(x^2) \leq G$. Since t inverts X, X is an abelian group without involutions.

14. Let G be a finite group with an involutive automorphism[5] α without nontrivial fixed points. Show that G is inverted by α.

15a. Let G be a group of prime exponent p. Show that for $g \in G^*$, no two distinct elements of $\langle g \rangle$ can be conjugated in G.

b. Show that if $\exp(G) = p$, then G has at least p conjugacy classes.

c. (Reineke) Let G be a group and assume that for some $x \in G$ of finite order, we have $G = x^G \cup \{1\}$. Show that $|G| = 1$ or 2.

16. Let G be an arbitrary torsion group without involutions. Show that G is 2-divisible. Assume G has an involutive automorphism α that does not fix any nontrivial elements of G. We will show that G is abelian and is inverted by α.

a. Show that for $a, b \in G$, if $a^2 = b^2$ then $a = b$.

Let $g \in G$. Let $h \in G$ be such that $h^2 = g^\alpha g$.

b. Show that $(h^\alpha)^2 = (h^{-1})^2$. Conclude that $h^\alpha = h^{-1}$.

c. Show that $(gh^{-1})^\alpha = gh^{-1}$. Deduce that $g = h$. This proves the result.

---

[3] An involution of a group is an element of order 2.

[4] I.e. has no elements of order 2.

[5] $\alpha \in \mathrm{Aut}(G)$ is an involutive automorphism if $o(\alpha) = 2$.