Group Theory

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Throughout, G is a group.

I. Basics.

1. Let H, $K \leq G$. Assume that for all $k \in K$, $kHk^{-1} \subseteq H$. Show that $kHk^{-1} \subseteq H$, i.e. that $K \leq N_G(H)$.
2. Show that a group of prime order is abelian.
3. Show that an abelian group is simple[1] iff it has prime order.
4. Classify all groups without nontrivial proper subgroups.
5. Let $g \in G$, $H \leq G$ and n and m two integers prime to each other.
   5a. Assume that g^m, $g^n \in H$. Show that $g \in H$.
   5b. Assume that $g^m = g^n = 1$. Show that $g = 1$.
6. If $|G|$ is finite and divisible by n, is it true that G necessarily has an element of order n?
7. Let G be a group
   7a. Show that if $A \subseteq B \subseteq G$, then $C_G(B) \leq C_G(A)$.
   7b Show that for any $A \subseteq G$, $A \subseteq C_G(C_G(A))$.
   7c. Show that for any $A \subseteq G$, $C_G(A) = C_G(C_G(C_G(A)))$.
8. Let H, K be normal subgroups of G. Show that if $H \cap K = 1$ then $hk = kh$ for all $h \in H$ and $k \in K$.
9. Let $H \lhd G$, $\overline{G} = G/H$ and $x \in G$. We know that $C_{\overline{G}}(\overline{x}) = C/H$ for some unique subgroup C containing H. Define C in terms of x and H.
10. Let $H \leq G$. Let G/H denote the left coset space. For $g \in G$ and $xH \in G/H$, let
$$g^*(xH) = gxH.$$
   10a. Show that $g^* \in \mathrm{Sym}(G/H)$.
   10b. Show that the map $g \rightarrow g^*$ is a homomorphism from G into $\mathrm{Sym}(G/H)$.
   10c. What is the kernel of the homomorphism *?
   10d. Assuming that $[G : H] = n < \infty$, show that $[G : \cap_{g \in G}\, g^{-1}Hg]$ divides $n!$.

II. Small Groups.

11. Show that a group of order 20 has a normal subgroup of order 5.
12. Show that groups of order 28 or 40 are not simple.
13. Let p and q be two distinct primes and G have order pq^n for some $n \geq 0$. Assume that $q > p$. Show that G has a normal subgroup A such that G/A is abelian.
14. Let p be a prime, m a natural number such that $(m, p) = 1$ and $m < p$. Let G have order pm. Show that G has a normal subgroup of order p.
15. Assume $G/Z(G)$ is cyclic. Show that G is abelian.
16. Let p be a prime and G have order p^n for some n.
   16a. Show that for any $g \in G$, $|g^G| = p^i$ for some $i = 0, 1, ..., n - 1$.
   16b. Conclude that $Z(G) \neq 1$.
   16c. Conclude that a group of order p^2 is necessarily abelian.

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[1] A group with no proper, nontrivial normal subgroups is called simple.

16d. Conclude that for any $i = 0, ..., n-1$, G has a normal subgroup of order p^i.

17. Assuming all the above, except may be for $n = 24, 36, 48$ and 56, a simple group of order $n < 60$ must be abelian.

18. Show that there are no simple groups of order 24, 36, 48 or 56.

III. Nilpotent Groups.

19. Let $Z_0(G) = 1$ and define

$$Z_{i+1}(G) = \{z \in G : g^{-1}z^{-1}gz \in Z_i(G) \text{ all } g \in G\}$$

inductively. Show that $Z_i(G)$ is a characteristic subgroup of G for all i. Conclude that $Z_i(G) \lhd G$. Show that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Show that if G is a finite p-group then $Z_k(G) = G$ for some k. Such a group is called nilpotent.

20. Assume G is nilpotent and let $1 \neq H \lhd G$. Show that $H \cap Z(G) \neq 1$.

21. Assume G is nilpotent and let $H < G$. Show that $H < N_G(H)$.