Throughout, $G$ is a group.

I. Basics.
1. Let $H, K \leq G$. Assume that for all $k \in K$, $kHk^{-1} \subseteq H$. Show that $kHk^{-1} \subseteq H$, i.e. that $K \leq N_G(H)$.
2. Show that a group of prime order is abelian.
3. Show that an abelian group is simple\(^1\) iff it has prime order.
4. Classify all groups without nontrivial proper subgroups.
5. Let $g \in G$, $H \leq G$ and $n$ and $m$ two integers prime to each other.
   5a. Assume that $g^m, g^n \in H$. Show that $g \in H$.
   5b. Assume that $g^m = g^n = 1$. Show that $g = 1$.
6. If $|G|$ is finite and divisible by $n$, is it true that $G$ necessarily has an element of order $n$?
7. Let $G$ be a group
   7a. Show that if $A \subseteq B \subseteq G$, then $C_G(B) \leq C_G(A)$.
   7b. Show that for any $A \subseteq G$, $A \subseteq C_G(C_G(A))$.
   7c. Show that for any $A \subseteq G$, $C_G(A) = C_G(C_G(C_G(A)))$.
8. Let $H, K$ be normal subgroups of $G$. Show that if $H \cap K = 1$ then $hk = kh$ for all $h \in H$ and $k \in K$.
9. Let $H \triangleleft G$, $\overline{G} = G/H$ and $x \in G$. We know that $C_{\overline{G}}(x) = C/H$ for some unique subgroup $C$ containing $H$. Define $C$ in terms of $x$ and $H$.
10. Let $H \leq G$. Let $G/H$ denote the left coset space. For $g \in G$ and $xH \in G/H$, let $g^*(xH) = gxH$.
   10a. Show that $g^* \in \text{Sym}(G/H)$.
   10b. Show that the map $g \to g^*$ is a homomorphism from $G$ into $\text{Sym}(G/H)$.
   10c. What is the kernel of the homomorphism $^*$?
   10d. Assuming that $[G : H] = n < \infty$, show that $[G : \cap_{g \in G} g^{-1}Hg]$ divides $n!$.

II. Small Groups.
11. Show that a group of order 20 has a normal subgroup of order 5.
12. Show that groups of order 28 or 40 are not simple.
13. Let $p$ and $q$ be two distinct primes and $G$ have order $pq^n$ for some $n \geq 0$. Assume that $q > p$. Show that $G$ has a normal subgroup $A$ such that $G/A$ is abelian.
14. Let $p$ be a prime, $m$ a natural number such that $(m, p) = 1$ and $m < p$. Let $G$ have order $pm$. Show that $G$ has a normal subgroup of order $p$.
15. Assume $G/Z(G)$ is cyclic. Show that $G$ is abelian.
16. Let $p$ be a prime and $G$ have order $p^n$ for some $n$.
   16a. Show that for any $g \in G$, $[g^i] = p^i$ for some $i = 0, 1, ..., n - 1$.
   16b. Conclude that $Z(G) \neq 1$.
   16c. Conclude that a group of order $p^2$ is necessarily abelian.

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\(^1\) A group with no proper, nontrivial normal subgroups is called simple.
16d. Conclude that for any \( i = 0, \ldots, n-1 \), \( G \) has a normal subgroup of order \( p^i \).

17. Assuming all the above, except may be for \( n = 24, 36, 48 \) and \( 56 \), a simple group of order \( n < 60 \) must be abelian.

18. Show that there are no simple groups of order 24, 36, 48 or 56.

III. Nilpotent Groups.

19. Let \( Z_0(G) = 1 \) and define
\[
Z_{i+1}(G) = \{ z \in G : g^{-1}z^{-1}gz \in Z_i(G) \text{ all } g \in G \}
\]
inductively. Show that \( Z_i(G) \) is a characteristic subgroup of \( G \) for all \( i \). Conclude that \( Z_i(G) \triangleleft G \). Show that \( Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)) \). Show that if \( G \) is a finite \( p \)-group then \( Z_k(G) = G \) for some \( k \). Such a group is called nilpotent.

20. Assume \( G \) is nilpotent and let \( 1 \neq H \triangleleft G \). Show that \( H \cap Z(G) \neq 1 \).

21. Assume \( G \) is nilpotent and let \( H < G \). Show that \( H \triangleleft N_G(H) \).