Group Theory

(Exercises on Commutators)

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G always stands for a group.

1. Show that "to be a characteristic subgroup" is a transitive relation. Conclude that $Z(G')$ is a characteristic subgroup of G.

2. Let $A \lhd G$. Find a condition so that $G/C_G(A)$ is an abelian group. Conclude that $G/C_G(Z(G'))$ is an abelian group.

3. Show that a minimal normal subgroup of a solvable group (in case it exists) is abelian.

4. Let G be a centerless group. Let A be a minimal normal subgroup of G contained in $Z(G')$. Show that for every $g \in G \setminus C_G(A)$, the set

$$\mathrm{ad}(g)(A) = \{[g, a] : a \in A\}$$

is either A or $\{1\}$.

5. Show that if H is an abelian group, then $\mathrm{Hom}(G, H) \approx \mathrm{Hom}(G/G', H)$.

6. Let A and B be two subgroups of G. Show that if $[[A, B], B] = 1$, then $[A, B]$ is abelian.

7. Show that if $G = G'H$ and $H \cap G' = 1$, then $[G, G'] = G'$.

8. Show that if G is generated by a subset S, then G' is the smallest normal subgroup containing $\{[s, t] : s, t \in S\}$.