Algebra (Math 222)

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Classwork on general linear groups

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Let K be a field. Define the **general linear group** over K as

$$\mathrm{GL}_n(K) = \{A \in \mathrm{M}_{n\times n} : A \text{ is invertible}\} = \{A \in \mathrm{M}_{n\times n}(K) : \det(A) \in K^*\}.$$

1. Show that $\mathrm{GL}_n(K)$ is a group.

2. Show that there is a one-to-one correspondance between $\mathrm{GL}_n(K)$ and the set of basis of the vector space K^n.

3. Show that $|\mathrm{GL}_n(\mathbf{F}_q)| = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$.

4. Show that $\mathrm{Z}(\mathrm{GL}_n(K)) = \{\lambda \mathrm{Id} : \lambda \in K^*\} \approx K^*$.

Define the **projective general linear group** $\mathrm{PGL}_n(K)$ over K as $\mathrm{GL}_n(K)/\mathrm{Z}(\mathrm{GL}_n(K))$.

5. Show that $|\mathrm{PGL}_n(\mathbf{F}_q)| = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})/(q-1)$.

Consider the group homomorphism $\det: \mathrm{GL}_n(K) \to K^*$. Note that the kernel of det is $\{A \in \mathrm{M}_{n\times n} : \det(A) = 1\}$. We denote this by $\mathrm{SL}_n(K)$ and call it the **special linear group** over K.

6. Show that det is onto.

7. Show that $\mathrm{SL}_n(K)$ is a normal subgroup of $\mathrm{GL}_n(K)$.

8. Show that $\mathrm{GL}_n(K)/\mathrm{SL}_n(K) \approx K^*$.

9. Show that

$$|\mathrm{SL}_n(\mathbf{F}_q)| = |\mathrm{PGL}_n(\mathbf{F}_q)| = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})/(q-1).$$

10. Show that $\mathrm{Z}(\mathrm{SL}_n(K)) = \mathrm{Z}(\mathrm{GL}_n(K)) \cap \mathrm{SL}_n(K) = \{\lambda\mathrm{Id} : \lambda^n = 1\}$.

Define the **projective special linear group** $\mathrm{PSL}_n(K)$ over K as $\mathrm{SL}_n(K)/\mathrm{Z}(\mathrm{SL}_n(K))$.

11. For a prime p find $|\mathrm{SL}_p(\mathbf{F}_q)|$. What can you say about $|\mathrm{SL}_n(\mathbf{F}_q)|$ for a general n?

12. Consider the composition α of the canonical homomorphisms

$$\mathrm{SL}_n(K) \to \mathrm{GL}_n(K) \to \mathrm{PGL}_n(K)$$

Show that $\mathrm{Ker}(\alpha) = \mathrm{Z}(\mathrm{SL}_n(K))$.

13. Conclude that there is a canonical injection from $\mathrm{PSL}_n(K)$ into $\mathrm{PGL}_n(K)$.

14. Find conditions on K for this injection to be onto (i.e. an isomorphism.)

15. Let B be the subgroup of upper triangular matrices of $\mathrm{GL}_2(K)$. Show that the keft coset space $\mathrm{GL}_n(K)/B$ is naturally in one-to-one correspondance with the set $K \cup \{\infty\}$ where ∞ is a new element not in K.

16. Show that if K is algebraically closed, then any element $g \in \mathrm{GL}_n(\mathbf{F}_q)$ is conjugate to an upper triangular matrix. (Hint: Jordan canonical form).

17. If $K \le L$, there is a natural injection of $\mathrm{GL}_n(K)$ into $\mathrm{GL}_n(L)$. Show that any any element $g \in \mathrm{GL}_n(\mathbf{F}_q)$ can be conjugated to an upper triangular matrix in $\mathrm{GL}_n(\mathbf{F}_{q'})$ for some q'. (Hint: Jordan canonical form).

18. Define $\mathrm{SL}_n(\mathbb{Z})$ in a similar way. For each $n \in \mathbb{Z}$, find a normal subgroup of index n^2 of $\mathrm{SL}_n(\mathbb{Z})$.

20. Can you show that $\mathrm{PSL}_n(K)$ is simple except for a few exceptions? (Reference: Suzuki or Scott, **Group Theory**).

21. Can $\mathrm{PSL}_n(\mathbf{F}_q)$ be isomorphic to $\mathrm{Alt}(m)$ for some m?

22. $SL_n(\mathbb{Z})$ is known to have a normal subgroup F which is a free subgroup generated by two elements such that $SL_n(\mathbb{Z})/F \approx$ Sym(3). Can you find it? (Reference: J.-P. Serre, **Trees**).