Group Theory

Math 311 First Midterm Ali Nesin November 1999

1. Let (G, X) be a transitive permutation group. A subset *Y* of *X* is called a **set** of imprimitivity if for all $g \in G$, either gY = Y or $gY \cap Y = \emptyset$. The set *X* and the singleton sets are sets of imprimitivity and they are called the trivial sets of imprimitivity. The permutation group (G, X) is called imprimitive if it has nontrivial imprimitive sets, otherwise the permutation group is called **primitive**. In other words, (G, X) is primitive, if $Y \subseteq X$ has the property that $\{gX : g \in G\}$ partition *X*, then either Y = X or |Y| = 1.

Show that (G, X) is primitive iff for some (equivalently all) $x \in X$, G_x is a maximal subgroup of G.

2. Let (G, X) be a doubly transitive permutation group. Let *B* be a one point stabilizer, i.e.,

$$B = G_x = \{g \in G : gx = x\}.$$

2a. Show that $G = B \cup BwB$ for all $w \in G \setminus B$.

2b. Conclude that B is a maximal subgroup of G and that G acts primitively on X.

2c. Without using any of the previous questions, show that G is a primitive permutation group. Conclude once more that B is a maximal subgroup of G.

2d. Show that if $1 \neq H \triangleleft G$, then G = HB.

2e. Conclude that a nontrivial normal subgroup of *G* acts transitively on *X*.

3. Show that, for $n \ge 5$, the only normal subgroups of Sym(n) are 1, Alt(n) and Sym(n).

4. Find an element of Aut(Sym(6)) \ Inn(Sym(6)).