Algebra

First Midterm (harder)
1999-2000
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1. Classify all groups of order 20.

2. Find all automorphisms of $\mathbf{Z}_{p^\infty}$.

3. Let $t \in G$ be an involution. Let $X = \{[t, g]: g \in G\}$

3a. Show that for $x \in X$, $x^t = x^{-1}$ and that $t \notin X$. Conclude that the elements of tX are involutions.

3b. Show that the map $\varphi : G/C_G(t) \to X$ defined by $\varphi(gC_G(t)) = [t, g^{-1}]$ is a well-defined bijection.

3c. Assume from now on that G is finite and that $C_G(t) = \{1, t\}$. We will show that X is an abelian 2'-subgroup and $G = X \rtimes \{1, t\}$. By part b, $|X| = |G|/2$. By part a and by assumption, X has no involutions. Therefore $X \cap tX = \varnothing$. Conclude that $G = X \sqcup tX$ and that X is the set of elements of order $\neq 2$ of G. Therefore, X is a characteristic subset of G. Let $x \in X \setminus \{1\}$ be a fixed element. Conclude that t^x inverts X as well (replace t by t^x). Conclude that $1 \neq x^2 = tt^x$ centralizes X. Therefore $X = C_G(x^2) \leq G$. Since t inverts X, X is an abelian group without involutions.

4. Let G be an arbitrary torsion group without involutions. Note that G is 2-divisible. Assume G has an involutive automorphism α that does not fix any nontrivial elements of G. We will show that G is abelian and is inverted by α.

4a. Show that for $a, b \in G$, if $a^2 = b^2$ then $a = b$.

Let $g \in G$. Let $h \in G$ be such that $h^2 = g^\alpha g$.

4b. Show that $(h^\alpha)^2 = (h^{-1})^2$. Conclude that $h^\alpha = h^{-1}$.

4c. Show that $(gh^{-1})^\alpha = gh^{-1}$. Deduce the result.

5. Show that in a finite group any two involutions[1] either are conjugate or commute with a third involution.

[1] An involution is an element of order 2.