Group Theory

Math 311
First Midterm
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1. Let (G, X) be a transitive permutation group. A subset Y of X is called a **set of imprimitivity** if for all $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$. The set X and the singleton sets are sets of imprimitivity and they are called the trivial sets of imprimitivity. The permutation group (G, X) is called **imprimitive** if it has nontrivial imprimitive sets, otherwise the permutation group is called **primitive**. In other words, (G, X) is primitive, if $Y \subseteq X$ has the property that $\{gX : g \in G\}$ partition X, then either $Y = X$ or $|Y| = 1$.

Show that (G, X) is primitive iff for some (equivalently all) $x \in X$, G_x is a maximal subgroup of G.

2. Let (G, X) be a doubly transitive permutation group. Let B be a one point stabilizer, i.e.,

$$B = G_x = \{g \in G : gx = x\}.$$

2a. Show that $G = B \cup BwB$ for all $w \in G \setminus B$.

2b. Conclude that B is a maximal subgroup of G and that G acts primitively on X.

2c. Without using any of the previous questions, show that G is a primitive permutation group. Conclude once more that B is a maximal subgroup of G.

2d. Show that if $1 \neq H \lhd G$, then $G = HB$.

2e. Conclude that a nontrivial normal subgroup of G acts transitively on X.

3. Show that, for $n \geq 5$, the only normal subgroups of $\mathrm{Sym}(n)$ are 1, $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$.

4. Find an element of $\mathrm{Aut}(\mathrm{Sym}(6)) \setminus \mathrm{Inn}(\mathrm{Sym}(6))$.