Group Theory Problems

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Throughout the exercises G is a group. We let $Z_i = Z_i(G)$ and $Z = Z(G)$.

Let H and K be two subgroups of finite index of G. Show that $H \cap K$ has also finite index in G. Show that H has finitely many conjugates in G. Conclude that if a group has a subgroup of finite index, then it has a normal subgroup of finite index. (See also Exercise 49).

1. (P. Hall)

a. Show that for $x, y, z \in G$, $[x, yz] = [x, z][x, y]^z$ and $[xy, z] = [x, z]^y[y, z]$. Conclude that if $H, K \leq G$, then H and K normalize the subgroup $[H, K]$. Conclude also that if $A \leq G$ is an abelian subgroup and if $g \in N_G(A)$, then $\mathrm{ad}(g) : A \to A$ is a group homomorphism whose kernel is $C_A(g)$.

b. Let x, y, z be three elements of G. Show that

$$[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1.$$

Conclude that if H and K are two subgroups of a group G and if $[[H, K], K] = 1$, then $[H, K'] = 1$.

c. (Three Subgroup Lemma of P. Hall) Let H, K, L be three normal subgroups of G. Using part b, show that $[[H, K], L] \leq [[K, L], H][[L, H], K]$.

d. Conclude from part (c) that

$$[G^i, G^j] \leq G^{i+j+1},$$
$$G^{(i)} \leq G^{2^i},$$
$$[G^i, Z_j] \leq Z^{j-i-1},$$
$$[Z^{i+1}, G^i] = 1.$$

e. Show that a nilpotent group is solvable. Show that the converse of this statement is false.

2. Let $A \lhd G$ be an abelian subgroup and let $g \in G$. By Exercise 1.a, $\mathrm{ad}(g): A \to A$ is a group homomorphism. Assume that $[G', A] = 1$. Show the following:

a. $C_A(g) \lhd G$.

b. $[g, A] = \mathrm{ad}(g)(A)$.

c. $[g, A] \lhd G$.

3a. Let G be nilpotent of class n. Show that $G^{n-i} \leq Z_i$. Conclude that $G = Z_n$.

b. Conversely, assume that $G = Z_n$. Show that $G^i \leq Z_{n-i}$. Conclude that G is nilpotent of class n.

c. Show that G is nilpotent of class n if and only if $Z_n = G$ and $Z_{n-1} \neq G$.

4. Let $H \lhd G$ and $K, L \leq G$.

a. Show that $[KH/H, LH/H] = [K, L]H/H$.

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[1] From Borovik-Nesin, "Groups of Finite Morley Rank", chapter 1.

b. Conclude that if G is solvable (resp. nilpotent), then so are H and G/H.

c. Show that if G/H and H are solvable, then so is G.

d. Find an example where the previous result fails if we replace the word "solvable" by "nilpotent".

e. Deduce from part c that if A and B are solvable subgroups of G and if one of them normalizes the other, then $\langle A, B \rangle = AB$ is also solvable.

5. Let $X \leq Z_n$ be a normal subgroup of G. Show that G is nilpotent if and only if G/X is. Let i be fixed integer. Show that G is nilpotent of class n if and only if G/Z_i is nilpotent of class $n - i$. Show that Z_i is nilpotent of class i. Find a (nilpotent) group where $Z_2 \neq Z$ and Z_2 is abelian. (See also Exercise 41).

6. Show that a nilpotent group G satisfies the **normalizer condition** (i.e. if $H < G$ then $H < N_G(H)$).

7. (Hirsch) Let G be a nilpotent group. Show that if $G = HN'$ for some $H \leq G$, then $H = G$.

8. (Hirsch) Let G be a nilpotent group. Show that if $1 \neq H \lhd G$, then $H \cap Z \neq 1$.

9. Let A and B be two normal nilpotent subgroups of G. Show that the subgroup $\langle A, B\rangle = AB$ is also normal and nilpotent.

10. a. Show that the subgroup G^n is generated by the elements of the form $[x_1, [x_2, ..., [x_n, x_{n+1}] ...]]$, where $x_i \in G$. Find a similar statement for $G^{(n)}$.

b. Show that an abelian group is locally finite if and only if it is a torsion group. Conclude that a solvable group is locally finite if and only if it is a torsion group.

c. Let p be a prime. Show that a nilpotent-by-finite p-group is solvable and hence locally finite.

11. Show that for $x, y \in G$ and n a positive integer, $[x^n, y] = [x, y]^{x^{n-1}}[x, y]^{x^{n-2}}...[x, y]$.

12a. Let $g \in G$ and $H \leq G$ be such that $[g, H] \subseteq Z$. Show that the map $\text{ad}(g)\colon H \rightarrow Z$ is a group homomorphism. Show that for all $h \in H$, $n \in Z$,

$[g, h]^n = [g^n, h] = [g, h^n]$.

b. Using Exercise 11, show that if $z \in Z_2$ and $z^n \in Z$, then $[z, G]$ is a central subgroup of finite exponent and that $\exp([z, G])$ divides n.

c. (Mal'cev, McLain) Use part b to prove, by induction on the nilpotency class, that if a nilpotent group has an element of order p where p is a prime, then it has central elements of order p.

d. Let G be a nilpotent group and D a p-divisible subgroup of G. Show that D commutes with all the p-elements of G. Deduce that in a divisible nilpotent group, elements of finite order form a central subgroup.

13. *p*-Divisible Nilpotent Groups. (Chernikov) Let p be a prime and let G be a p-divisible nilpotent group.

a. Show that if $g^p \in Z$, then $g \in Z$.

b. Conclude that Z is p-divisible, contains all the p-elements and that G/Z is p-torsion-free and p-divisible.

c. Show that G/Z_i is p-torsion-free for all $i \geq 1$.

d. Conclude that Z_{i+1}/Z_i is p-torsion-free and p-divisible for $i \geq 1$.

14. Let G be a nilpotent group.

a. Let $i \geq 1$ be an integer. Show that G/G^i is p-divisible if and only if G/G^{i+1} is p-divisible.

b. Conclude that G is p-divisible if and only if G/G' is p-divisible.

c. Show that G has a unique maximal p-divisible subgroup D.

d. Assume that for some $D \lhd G$, D and G/D are p-divisible. Show that G is p-divisible.

15. (Dixmier). Let G be nilpotent and assume that $\exp(G/G') = n$.

a. Show that $\exp(G^i/G^{i+1}) \mid n$ for all i.

b. Conclude that $\exp(G) \mid n^c$ where c is the nilpotency class of G.

16. Let P be a Sylow p-subgroup of G. Show that P is characteristic in $N_G(P)$. Conclude that $N_G(N_G(P)) = N_G(P)$. By Exercise 6, if G is nilpotent, $N_G(P) = G$, i.e. $P \lhd G$.

Conclude that, for a given prime p, a nilpotent group G has a unique Sylow p-subgroup, and that if G is torsion, then G is the direct sum of its Sylow p-subgroups.

17. Let $t \in G$ be an involution. Let $X = \{[t, g]: g \in G\}$

a. Show that for $x \in X$, $x^t = x^{-1}$ and that $t \notin X$. Conclude that the elements of tX are involutions.

b. Show that the map $\varphi : G/C_G(t) \to X$ defined by $\varphi(gC_G(t)) = [t, g^{-1}]$ is a well-defined bijection.

c. Assume from now on that G is finite and that $C_G(t) = \{1, t\}$. We will show that X is an abelian 2'-subgroup and $G = X \rtimes \{1, t\}$. By part b, $|X| = |G|/2$. By part a and by assumption, X has no involutions. Therefore $X \cap tX = \varnothing$. Conclude that $G = X \sqcup tX$ and that X is the set of elements of order $\neq 2$ of G. Therefore, X is a characteristic subset of G. Let $x \in X \setminus \{1\}$ be a fixed element. Conclude that t^x inverts X as well (replace t by t^x). Conclude that $1 \neq x^2 = tt^x$ centralizes X. Therefore $X = C_G(x^2) \leq G$. Since t inverts X, X is an abelian group without involutions.

18. Let G be a finite group with an involutive automorphism α without nontrivial fixed points. Show that G is inverted by α.

19a. Let G be a group of prime exponent p. Show that for $g \in G^*$, no two distinct elements of $\langle g \rangle$ can be conjugated in G.

b. Show that if $\exp(G) = p$, then G has at least p conjugacy classes.

c. (Reineke) Let G be a group and assume that for some $x \in G$ of finite order, we have $G = x^G \cup \{1\}$. Show that $|G| = 1$ or 2.

20. Let G be an arbitrary torsion group without involutions. Note that G is 2-divisible (see Exercise 36). Assume G has an involutive automorphism α that does not fix any nontrivial elements of G. We will show that G is abelian and is inverted by α.

a. Show that for $a, b \in G$, if $a^2 = b^2$ then $a = b$.

Let $g \in G$. Let $h \in G$ be such that $h^2 = g^\alpha g$.

b. Show that $(h^\alpha)^2 = (h^{-1})^2$. Conclude that $h^\alpha = h^{-1}$.

c. Show that $(gh^{-1})^\alpha = gh^{-1}$. Deduce that $g = h$. This proves the result.

21. (I. Schur[2]). Assume that G/Z is finite. We will show that G' is finite. Let $|G/Z| = n$.

a. Show that the set $X = \{[g, h]: g, h \in G\}$ has cardinality n^2.

b. Let $X = \{x_1, ..., x_k\}$. Show that $G' = \{ x_1^{n_1} ... x_k^{n_k} : n_i \in \mathbf{N}\}$.

c. Show that for all $g, h \in G$,

$[g, h]^{n+1} = g^{-1}[g, h]^n g[g, h] = g^{-1}[g, h]^{n-1}\left[g^2, h\right]^{g^{-1}} g.$

d. Conclude from parts (b) and (c) that every element of G' is a product of at most n^3 elements of X and so G' is finite.

22. (R. Baer). Let A, B be subgroups of G that normalize each other. Assume that the set $X = \{[a, b]: a \in A, b \in B\}$ is finite. We will show that $[A, B]$ is finite. Note first that, without loss of generality, we may assume that $G = AB$. With this assumption A and B are normal subgroups of G. Let $U = [A, B] \leq A \cap B$. Clearly $U \lhd G$.

a. Show that $C_G(X)$ is a normal subgroup of finite index in G. Show that $C_G(X)$ centralizes U.

b. Deduce from part (a) that $C_G(X) \cap U$ is a central subgroup of U and has finite index in U. Exercise 21 implies that U' is finite.

c. Show that, without loss of generality, we may assume that $U' = 1$.

d. Clearly the subset $\{[a, u] : a \in A, u \in U\}$ of X is finite and these elements commute with each other. Show that $[a, u]^2 = [a, u^2]$. Conclude that $[A, U]$ is finite. Show that, without loss of generality, we may assume that $[A, U] = 1$. Conclude that, without loss of generality U is central in G.

e. Show that X is closed under the squaring map $x \mapsto x^2$. Conclude that $[A, B]$ is finite.

23. Completely Reducible Groups. A group is said to be **completely reducible** if it is the direct sum of finitely many nonabelian simple groups. A subgroup H of a group G is called **subnormal** if there is a finite chain $H = H_1 \lhd ... \lhd H_n = G$. We assume in this exercise that G is a completely reducible group and we let $G = \oplus_{i=1,...,n} A_i$ where each A_i is a simple nonabelian group.

a. We first want to show that any simple, nontrivial and normal subgroup H of G is one of the subgroups A_i. This will show that the subgroups A_i are uniquely determined. Let $1 \neq h = a_1 ... a_n \in H$ where $a_i \in A_i$. Assume $a_i \neq 1$. Let $b_i \in A_i \setminus \mathbf{C}_{A_i}(a_i)$. Show that $1 \neq [h, b_i] \in A_i \cap H$, conclude that $A_i = H$.

b. Show that every normal subgroup of G is a direct sum of the subgroups A_i. Conclude that every normal subgroup of G is completely reducible and has a complement.

[2] If I am not mistaken, this exercise, the way it intends to lead to the result, contains a mistake.

c. Conclude from (a) that a subnormal subgroup of G is a normal subgroup.

24. Let G be a finite group. Let A be a minimal normal subgroup of G.

a. Show that if A has a nontrivial normal solvable subgroup, then A is an elementary abelian subgroup.

From now on we assume that A has no nontrivial normal solvable subgroup. We will show that A is completely reducible. Let B be a minimal A-normal subgroup of A.

b. Show that for any $g \in G$, B^g is also a minimal A-normal subgroup of A. Conclude that if $C \lhd A$, then either $B^g \leq C$ or $C \cap B^g = 1$. Deduce that $A = B^{g_1} \oplus \ldots \oplus B^{g_n}$ for some $g_1, \ldots, g_n \in G$.

c. Show that B is simple.

25. (Generalized Quaternions) Let G be the group generated by the elements x and y subject to the relations $x^m = y^2$ and $x^y = x^{-1}$ where $m > 0$. Show that $\langle x \rangle \lhd G$. Note that $x^{-m} = x^{my} = y^{2y} = y^2 = x^m$. Conclude that $|G| \leq 4m$. In fact $|G| = 4m$. When m is even, G is called a **generalized quaternion group**. When $m = 2$, G is called the **quaternion group**.

Semidirect Products

Let U and T be two groups and let φ: $T \to \text{Aut}(U)$, $t \to \varphi_t$ be a group homomorphism. We will construct a new group denoted by $U \rtimes_\varphi T$, or just by $U \rtimes T$ for short. The set on which the group operation is defined is the Cartesian product $U \times T$, and the operation is defined as follows: $(u, t)(u', t') = (u.\varphi_t(u'), tt')$. The reader will have no difficulty in checking that this is a group with (1, 1) as the identity element. The inverse is given by the rule: $(u, t)^{-1} = (\varphi_{t^{-1}}(u^{-1}), t^{-1})$. Let G denote this group. G is called the semidirect product of U and T (in this order; we also omit to mention φ). U can be identified with $U \times \{1\}$ and hence can be regarded as a normal subgroup of G. T can be identified with $\{1\} \times T$ and can be regarded as a subgroup of G. Then the subgroups U and T of G have the following properties: $U \lhd G$, $T \leq G$, $U \cap T = 1$ and $G = UT$.

Conversely, whenever a group G has subgroups U and T satisfying these properties, G is isomorphic to a semidirect product $U \rtimes_\varphi T$ where $\varphi : T \to \text{Aut}(U)$ is given by $\varphi_t(u) = tut^{-1}$.

When $G = U \rtimes T$, one says that the group G is **split**[3]; then the subgroups U and T are called each other's **complements**. We also say that T (or U) splits in G. Note that T is not the only complement of U in G: for example, any conjugate of T is still a complement of U.

When the subgroup U is abelian, it is customary to denote the group operation of U additively. In this case, it is suggestive to let $tu = \varphi_t(u)$. Then the group operation can be written as: $(u, t)(u', t') = (tu' + u, tt')$. The reader should compare this with the following formal matrix multiplication:

$$\begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t' & u' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} tt' & tu'+u \\ 0 & 1 \end{pmatrix}$$

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[3] This is an abuse of language: every group G is split, for example as $G = G \rtimes \{1\}$. When we use the term "split", we have either U or T around.

Examples.

1. Let V be a vector space and GL(V) be the group of all vector space automorphisms of V. The group $V \rtimes \mathrm{GL}(V)$ (where $\varphi = \mathrm{Id}$) is a subgroup of Sym(V) as follows: $(v, g)(w) = gw + v$.

2. The subgroup $B_n(K)$ that consists of all the invertible $n \times n$ upper triangular matrices over a field K is the semidirect product of $\mathrm{UT}_n(K)$ (upper-triangular matrices with ones on the diagonal) and $\mathrm{T}_n(K)$ (invertible diagonal matrices).

Exercises.

26. Let K be any field. Show that the group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*,\ u \in K \right\}$$

is a semidirect product of the form $G' \rtimes T$ for some subgroup T. This group is called the **affine group**.

27. Show that the direct product of two groups is a special case of semidirect product.

28. Let $G = U \rtimes T$.

a. Let $U \le \mathrm{H} \le G$. Show that $H = U \rtimes (\mathrm{H} \cap \mathrm{T})$.

b. Let $T \le \mathrm{H} \le G$. Show that $H = (\mathrm{U} \cap \mathrm{H}) \rtimes T$.

c. Show that if T is abelian then $G' \le U$.

d. Show that if $T_1 \le T$, then $\mathrm{N}_U(T_1) = \mathrm{C}_U(T_1)$.

29. Let $G = U \rtimes T$. Let $t \in T$ and $x \in U$. Show that xt is G-conjugate to an element of T if and only if xt is conjugate to t if and only $(xt)^u = t$ for some $u \in U$ if and only if $x \in [U, t^{-1}]$.

30. Let $G = U \rtimes T$ and let $V \le U$ be a G-normal subgroup of U. Show that $G/V \approx U/V \rtimes T$ in a natural way.

31. Let $G = U \rtimes T$ and let $V \le U$ be a G-normal subgroup of U. By Exercise 30, $G/V \approx U/V \rtimes T$. Let $t \in T$ be such that $V = \mathrm{ad}(t)(V)$ and $U/V = \mathrm{ad}(t)(U/V)$. Show that $U = \mathrm{ad}(t)(U)$.

32. Let K be a field and let n be a positive integer. For $t \in K^*$ and $x \in K$, let $\varphi_t(x) = t^n x$. Set $G = K^+ \rtimes_\varphi K^*$. What is the center of G? Show that $\mathrm{Z}_2(G) = \mathrm{Z}(G)$. What is the condition on K that insures $G' \approx K^+$? Show that G is isomorphic to a subgroup of $\mathrm{GL}_2(K)$.

Abelian Groups

We will need the following fact several times:

Fact 1. *Let G be an abelian group. Let D be a divisible subgroup of G. Then D has a complement in G, i.e. G = D ⊕ H for some H   G. Furthermore every subgroup disjoint from D can be extended to a complement of D.*

Sketch of the proof: It is enough to prove the second statement. Let *K* be a subgroup disjoint from *D*. Using Zorn's Lemma, find a subgroup *H* containing *K*, disjoint from *D* and maximal for these properties. The maximality of *H* insures that $G = D \oplus H$.

From this fact it follows that, for some subgroup *H*, $G = \mathrm{D}(G) \oplus H$ where $D(G)$ is the unique maximal divisible subgroup of *G*. Clearly *H* has no nontrivial, divisible subgroups.

We will also make use of the following elementary result:

Fact 2. *A finitely generated abelian group is a direct sum of finitely many cyclic groups.*

Prüfer *p*-group. Let *p* be any prime and consider the subset

$$\mathbf{Z}_{p^\infty} = \{x \in \mathbf{C} \colon x^{p^n} = 1 \text{ for some } n \in \mathbf{N}\}$$

of complex numbers of norm 1. With the usual multiplication of complex numbers, $\mathbf{Z}_{p^\infty}$ is an infinite countable abelian group. It is called the **Prüfer *p*-group**. Every element of $\mathbf{Z}_{p^\infty}$ has finite order p^n for some *n*. Given a natural number *n*, there are exactly p^n elements of $\mathbf{Z}_{p^\infty}$ that satisfy the equation $x^{p^n} = 1$ (namely the elements $e^{2k\pi i/p^n}$ where $k = 0, ..., p^n - 1$).

Note that $\mathbf{Z}_{p^\infty}$ is the union of the ascending chain of finite subgroups

$$\{x \in \mathbf{C} \colon x^{p^n} = 1\}$$

which are isomorphic to the cyclic groups $\mathbf{Z}/p^n\mathbf{Z}$. Thus every finite subset of $\mathbf{Z}_{p^\infty}$ generates a finite cyclic group (isomorphic to $\mathbf{Z}/p^n\mathbf{Z}$ for some $n \in \mathbf{N}$), i.e. $\mathbf{Z}_{p^\infty}$ is a locally cyclic group.

Exercises.

33. Let *G* and *H* be two abelian groups of the same prime exponent *p* (such groups are called **elementary abelian *p*-groups**) and of the same cardinality. Noting the fact that *G* and *H* are vector spaces over the field $\mathbf{F}_p$, show that these groups are isomorphic to each other. (See also Exercise 34).

34. Let *G* and *H* be two torsion-free abelian divisible groups of the same uncountable cardinality. Noting the fact that *G* and *H* are vector spaces over **Q**, show that $G \approx H$. (Compare this with Exercise 35).

35. Show that the group **Q** has no proper, nontrivial divisible subgroups. Conclude that **Q** and $\mathbf{Q} \oplus \mathbf{Q}$ are not isomorphic. Generalize this to $\oplus_{i=1,\ldots,n}\ \mathbf{Q}$.

36. Show that a (not necessarily abelian) torsion group that has no elements of order *p* where *p* is a prime is *p*-divisible. Show that a group which is *p*-divisible for all primes *p* is divisible. Deduce that $\mathbf{Z}_{p^\infty}$ is a divisible abelian group.

37. Show that if a divisible abelian group contains an element of order p, then it contains a subgroup isomorphic to $\mathbf{Z}_{p^\infty}$.

38. Let G be a divisible abelian group and p a prime. Let G_p be the set of elements of order p of G together with 1. G_p is an elementary abelian p-group and so it can be considered as a vector space over the field $\mathbf{F}_p$ of p elements. Let κ be the dimension of G_p over $\mathbf{F}_p$. Show that G contains a direct sum of κ copies of $\mathbf{Z}_{p^\infty}$. κ is called the **Prüfer *p*-rank** of G.

39. Let G be a divisible abelian group. Show that $G = \mathrm{T}(G) \oplus F$ where $T(G)$ is the set of torsion elements and F is some divisible torsion-free subgroup. Conclude that a divisible abelian group is isomorphic to a group of the form:

$$\oplus_{p\,\text{prime}} (\oplus_{I_p} \mathrm{Z}_{p^\infty}) \oplus (\oplus_I \mathrm{Q})$$

for some sets I_p and I.

40. Let K be an algebraically closed field. First assume that char(K) = 0. Show that

$$K^* = \oplus_{p\,\text{prime}} \mathrm{Z}_{p^\infty} \oplus (\oplus_I \mathrm{Q})$$

for some I. Now assume that char(K) = $p > 0$. Show that

$$K^* = \oplus_{q \neq p,\, q\ \text{prime}} \mathrm{Z}_{p^\infty} \oplus (\oplus_I \mathrm{Q})$$

for some I.

41. Let $G = \mathbf{Z}_{p^\infty} \rtimes \mathbf{Z}/2\mathbf{Z}$ where $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{Z}_{p^\infty}$ by inversion (i.e. if $1 \neq i \in \mathbf{Z}/2\mathbf{Z}$ then $\varphi_i(g) = g^{-1}$ for all $g \in \mathbf{Z}_{p^\infty}$. Show that G is solvable of class 2, nonnilpotent but that the chain $(Z\_n(G))_{n \in \mathbf{N}}$ is strictly increasing. Show that G is isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_2(K)$ where K is an algebraically closed field of characteristic $\neq 2$. (Recall that $\mathrm{SL}_2(K)$ is the group consisting of 2×2 matrices of determinant 1 over K, and $\mathrm{PSL}_2(K)$ is the factor group of $\mathrm{SL}_2(K)$ modulo its center that consists of the two scalar matrices ± 1). What is the Sylow 2-subgroup of $\mathrm{SL}_2(K)$ when char(K) = 2?

42. Let G be the direct sum of finitely many copies of $\mathbf{Z}_{p^\infty}$. Show that if $H \leq G$ is an infinite subgroup then H contains a nontrivial divisible subgroup.

43. This exercise will show the advantages of the additive notation over the multiplicative one. Let G be a group and let $A \leq G$ be an abelian subgroup. Let $g \in \mathrm{N}_G(A)$. Thus g acts on A by conjugation. Let $\breve{g} \in \mathrm{Aut}(A)$ denote the automorphism of A induced by g. We can view $\breve{g}$ as an element of the ring End(A) and, denoting A additively, we can consider the endomorphism $\breve{g} - 1$. (For $a \in A$, $(\breve{g} - 1)(a)$ translates into $[g, a^{-1}]$ when the group operation of A is denoted multiplicatively). Let p be a prime number and assume that $g^p \in \mathrm{C}_G(A)$. Show that either $g \in \mathrm{C}_G(A)$ or $\breve{g}$ is an automorphism of order p. Assume now that $\exp(A) = p$. Show that $(\breve{g} - 1)^p = 0$. Conclude that $\mathrm{C}_A(g) \neq 0$. Conclude also that if A is infinite then $\mathrm{C}_A(g)$ is also infinite.

44a. Conclude from the preceding exercise that if $H \lhd G$ is a normal subgroup of finite index with a nontrivial center and if G is a p-group for a prime p, then G has a nontrivial center. Deduce that a nilpotent-by-finite p-group has a nontrivial center.

b. Let G be a nilpotent-by-finite p-group. Let $1 \neq H \lhd G$. Show that $H \cap Z(G) \neq 1$.

c. Show that if G is a nilpotent-by-finite p-group and $X < G$, then $X < N_G(X)$. This property is called the **normalizer condition**.

Permutation Groups.

Let G be a group and X a set. We say that G **acts** on X or that (G, X) is a **permutation group** if there is a map $G \times X \to X$ (denoted by $(g, x) \to g*x$ or gx) that satisfies the following properties:

1 For all $g, h \in G$ and all $x \in X$, $g(hx) = (gh)x$.

2 For all $x \in X$, $1x = x$.

This is saying that there is a group homomorphism $\varphi: G \to \mathrm{Sym}(X)$ where $\mathrm{Sym}(X)$ is the group of all bijections of X. The kernel of φ is called the **kernel** of the action. When φ is one-to-one, the action is called **faithful**. In other words, G acts faithfully on X when $gx = x$ for all $x \in X$ implies $g = 1$. Note that $G/\ker(\varphi)$ acts on X in a natural way: $\breve{g}x = gx$, and this action is faithful.

Two permutation groups (G, X) and (H, Y) are called **equivalent** if there are a group isomorphism $f : G \to H$ and a bijection $\varphi : X \to Y$ such that for all $g \in G$, $x \in X$ we have $\varphi(gx) = f(g)\varphi(x)$.

Let (G, X) be a permutation group. For any $Y \subseteq X$, we let

$$G_Y = \{g \in G : gy = y \text{ for all } y \in Y\}.$$

G_Y is called the **pointwise stabilizer** of Y. Note that $G_Y \leq G$ is a subgroup. When $Y = \{x_1, ..., x_n\}$, we write $G_{x_1,...,x_n}$ instead of G_Y. Clearly G_Y is the intersection of the subgroups G_y for $y \in Y$.

For $g \in G$ and $Y \subseteq X$ we define $gY = \{gy: y \in Y\}$ and the **setwise stabilizer** $G(Y) = \{g \in G : gY = Y\}$ of Y. We have $G_Y \leq G(Y)$. Finally for $A \subseteq G$, we define

$$F(A) = \{x \in X: ax = x \text{ for all } a \in A\},$$

the set of fixed points of A.

Exercise.

45. Let $A, B \subseteq G$ and $Y, Z \subseteq X$. Then the following hold:

i. $A \subseteq G_{F(A)}$.

ii. $Y \subseteq F(G_A)$.

iii. If $A \subseteq B$ then $F(B) \subseteq F(A)$.

iv. If $Y \subseteq Z$, then $G_Z \leq G_Y$.

v. $F(G_{F(A)}) = F(A)$.

vi. $G_{F(G_Y)} = G_Y$.

We say that G acts ***n*-transitively** on X if $|X| \geq n$ and if for any pairwise distinct $x_1, ..., x_n \in X$ and any pairwise distinct $y_1, ..., y_n \in X$, there is a $g \in G$ such that $gx_i = y_i$ for all $i = 1, ..., n$. **Transitive** means 1-transitive. We say that (G, X) is **sharply *n*-transitive** if it is n-transitive and if the stabilizer of n distinct points is reduced to $\{1\}$; in other words, if for any distinct $x_1, ..., x_n \in X$ and any distinct $y_1, ..., y_n \in X$, there is a unique $g \in G$ such that $gx_i = y_i \in X$ for all $i = 1, ..., n$. Sharply 1-transitive actions are also called **regular actions**. Up to equivalence, each group has only one regular action (see Exercise 46). Clearly, for every n and $|X| = n$, (Sym(X), X) is sharply n and also sharply (n–1)-transitive. If for $g \in G$, $x \in X$, $gx = x$ implies $g = 1$, we say that the action of G is **free** or that G acts **freely** on X.

Let X be a group and $G \leq \mathrm{Aut}(X)$. Then (G, X) is a permutation group. By abuse of language, one says that G acts **freely** (resp. **regularly**) on X if G acts freely (resp. regularly) on X^*.

Now we give the most important and, up to equivalence, the only example of transitive group actions:

Left-Coset Representation. Let G be a group and $B \leq G$ a subgroup. Set $X = G/B$, the left-coset space. We can make G act on X by left multiplication: $h(gB) = hgB$. This action is called the **left-coset action**, or the the **left-coset representation**. The kernel of this action is the core $\cap_{g\in G} B^g$ of B in G, which is the maximal G-normal subgroup of B.

Exercises

46. Let (G, X) be a transitive permutation group. Let $x \in X$ be any point and let $B = G_x$. Then the permutation group (G, X) is equivalent to the left-coset representation (G, G/B). (Hint: Let $f = \mathrm{Id}_G$ and φ: $G/B \to X$ be defined by $\varphi(gB) = gx$.)

47. If $N_G(B) = B$, then the left-coset action of G on G/B is equivalent to the conjugation action of G on $\{B^g : g \in G\}$.

48. Let (G, X) be a 2-transitive group and $B = G_x$. Then $G = B \sqcup BgB$ for every $g \in G \setminus B$. In particular B is a maximal subgroup of G. Conversely, if G is a group with a proper subgroup B satisfying the property $G = B \cup BgB$ for every (equivalently some) $g \in G \setminus B$, then the permutation group (G, G/B) is 2-transitive. (Hint: Assume G is 2-transitive, and let x and B as in the statement. Let $g \in G \setminus B$ be a fixed element of G. Let $h \in G \setminus B$ be any element. Since G is 2-transitive, there is an element $b \in G$ that sends the pair of distinct points (x, gx) to the pair of distinct points (x, hx). Thus $b \in B$ and $bgx = hx$, implying $h^{-1}bg \in B$ and $h \in BgB$.)

49. Let G be a group and let $H \leq G$ be a subgroup. Assume $[G{:}H] = n$. By considering the coset action $G \to \mathrm{Sym}(G/H)$ show that $[G : \cap_{g\in G} H^g]$ divides $n!$. The subgroup $\cap_{g\in G} H^g$ is called the **core** of H in G.

50. Let (G, X) be a permutation group. Assume G has a regular normal subgroup A (i.e. the permutation group (A, X) is regular). Show that $G = A \rtimes G_x$ for any $x \in X$. Show that (G, X) is equivalent to the permutation group (G, A) where $G = A \rtimes G_x$ acts on A as follows: For $a \in A$, $h \in G_x$ and $b \in A$, $(ah).b = ab^{h^{-1}}$. Show that G is faithful if and only if $C_H(A) = 1$.

51 Let (G, X) be a permutation group. Show that $G_{g^{-1}x} = G_x{}^g$ for any $x \in X$. Show that if G is an n-transitive group, then for any $1 \leq i \leq n$, all the i-point stabilizers are conjugate to each other.

52. Let (G, X) be a transitive permutation group. Show that if G is abelian then, for any $x \in X$, G_x is the kernel of the action and $(G/G_x, X)$ is a regular permutation group.

53. Let $n \geq 2$ be an integer. Show that (G, X) is n-transitive if and only if $(G_x, X \setminus \{x\})$ is $(n-1)$-transitive for any (equivalently some) $x \in X$. State and prove a similar statement for sharply n-transitive groups.

54. Let (G, X) be a permutation group. A subset $Y \subseteq X$ is called a **set of imprimitivity** if for all $g, h \in G$, either $gY = hY$ or $gY \cap hY = \emptyset$. If the only sets of imprimitivity are the singleton sets and X, then (G, X) is called a **primitive** permutation group. Show that a 2-transitive group is primitive. Assume that (G, X) is transitive. Show that (G, X) is primitive if and only if G_x is a maximal subgroup for some (equiv. all) $x \in X$. Conclude that if G is a 2-transitive group, then G_x is a maximal subgroup. (This also follows from Exercise 48).

55. Let G be a group and $B < G$ be a proper subgroup with the following properties: There is a $g \in G$ such that $G = B \cup BgB$ and if $agb = a'gb'$ for $a, a', b, b' \in B$ then $a = a'$ and $b = b'$. Show that $(G, G/B)$ is a sharply 2-transitive permutation group.

56. Let $G = A \rtimes H$ be a group where H acts regularly on A by conjugation (i.e. on A^*). Show that G is a sharply 2-transitive group.

57. Let (G, X) be a sharply 2-transitive permutation group, and for a fixed $x \in X$, set $B = G_x$. Show that for any fixed $g \in G \setminus B$, $G = B \sqcup BgB$ and if $agb = a'gb'$ for $a, b, a', b' \in B$, then $a = a'$ and $b = b'$. Show also that the conjugates of B are disjoint from each other. Show that there are involutions that swap given any two points. Conclude that there are involutions outside of B.

58. Show that the group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*,\ u \in K \right\}$$

acts sharply 2-transitively on the set

$$X = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in K \right\}.$$

59. Show that $G = PGL_2(K) = GL_2(K)/Z$ where Z is the set of scalar matrices (which is exactly the center of $GL_2(K)$) acts sharply 3-transitively on G/B where $B = B_2(K)$. Show that there is a natural correspondence between G/B and the set $K \cup \{\infty\}$. Transport the action of G on $K \cup \{\infty\}$ and describe it algebraically.

60. Let V be a vector space over a field K. Show that $V \rtimes GL(V)$ acts 2-transitively on V (see Example 1). Show that, when $\dim_K(V) = 1$, we find the example of Exercise 58.