Math 212
Midterm 1
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Notation: Let $G$ be a group, $x, y \in G$ be two elements of $G$ and $p$ be a positive integer (mainly a prime number). We define,

$$Z(G) = \{z \in G : zg = gz\}$$
$$x^r = y^{-1}xy$$
$$x^G = \{x^g : g \in G\}$$
$$C_G(x) = \{c \in G : cx = xc\}$$

A $p$-group is a group whose elements have order powers of $p$.
A $p'$-group is a group whose elements have finite orders prime to $p$.

A group $G$ is called divisible if, for every $g \in G$ and $n \in \mathbb{N} \setminus \{0\}$, there is an element $h \in G$ such that $h^n = g$.

A group $G$ is called $p$-divisible if, for every $g \in G$ there is an element $h \in G$ such that $h^p = g$.

I. Let $G$ be a group with two disjoint normal subgroups $A$ and $B$. Show that $ab = ba$ for all $a \in A$ and $b \in B$.

II. Let $p$ be a prime and let $G$ be a group with a normal $p'$-subgroup $H$ such that $G/H$ is cyclic of order $p$. The purpose of this exercise is to show that $G = HK$ for some subgroup $K$ of $G$ for which $H \cap K = 1$.

II.1. Show that a $p'$-group is $p$-divisible. (Hint: First show it for finite abelian groups.)

II.2. Show that there exists an element $x \in G \setminus H$ such that $x^p = 1$.

II.3. Let $x$ be as above and let $K = \langle x \rangle$. Show that $G = HK$ and $H \cap K = 1$.

III. Let $G$ be a group of order $pq$ where $p$ and $q$ are two distinct primes. Let $H$ be a Sylow $p$-subgroup and $K$ a Sylow $q$-subgroup of $G$. Clearly $H \cap K = 1$.

III.1. Show that $G = HK = KH$.

III.2. Show that if $p$ does not divide $q - 1$, then $H$ is normal.

III.3. Show that $H$ is normal.

III.4. Show that if $q$ does not divide $p - 1$ then $G$ is abelian. (Hint: Use III.1 and I).

For $k \in K$, let $k^*$ denote the function from $H$ into $H$ given by $k^*(h) = khk^{-1}$.

III.5. Show that for all $k \in K$, $k^* \in \text{Aut}(H)$.

III.6. Show that the map $*$ is a homomorphism from $K$ into $\text{Aut}(H)$.

III.7. Show that the map $*$ determines $G$ up to isomorphism.

III.8. Show that, up to isomorphism, there are at most $(p, q - 1)$ groups of order $pq$.

IV. Let $G$ be a group.

IV.1. Show that if $G/Z(G)$ is cyclic, then $G$ is abelian.

IV.2. For $x \in G$, show that there is a one-to-one correspondance between the conjugacy class $x^G$ and the right coset space $G/C_G(x)$.
IV.3. Suppose $G$ is finite. Show that

$$|G| = |Z(G)| + \sum_{x \in X} |G/C_G(x)|$$

for some subset $X$ of $G \setminus Z(G)$.

IV.4. Conclude that a finite nontrivial $p$-group has a nontrivial center.

IV.5. Conclude that a group of order $p^2$ is abelian.

IV.6. Classify all groups of order $\leq 7$.

V. Let $G$ be an abelian group and $H$ a divisible subgroup of $G$. Show that

$$G = H \oplus K$$

for some subgroup $K$ of $G$. 