Math 212

Midterm 1
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Notation: Let G be a group, $x, y \in G$ be two elements of G and p be a positive integer (mainly a prime number). We define,

$$Z(G) = \{z \in G : zg = gz\}$$
$$x^y = y^{-1}xy$$
$$x^G = \{x^g : g \in G\}$$
$$C_G(x) = \{c \in G : cx = xc\}$$

A ***p*-group** is a group whose elements have order powers of p.

A ***p′*-group** is a group whose elements have finite orders prime to p.

A group G is called **divisible** if, for every $g \in G$ and $n \in \mathbf{N} \setminus \{0\}$, there is an element $h \in G$ such that $h^n = g$.

A group G is called ***p*-divisible** if, for every $g \in G$ there is an element $h \in G$ such that $h^p = g$.

I. Let G be a group with two disjoint normal subgroups A and B. Show that $ab = ba$ for all $a \in A$ and $b \in B$.

II. Let p be a prime and let G be a group with a normal p'-subgroup H such that G/H is cyclic of order p. The purpose of this exercise is to show that $G = HK$ for some subgroup K of G for which $H \cap K = 1$.

II.1. Show that a p'-group is p-divisible. (**Hint:** First show it for finite abelian groups.)

II.2. Show that there exists an element $x \in G \setminus H$ such that $x^p = 1$.

II.3. Let x be as above and let $K = \langle x \rangle$. Show that $G = HK$ and $H \cap K = 1$.

III. Let G be a group of order pq where p and q are two distinct primes. Let H be a Sylow p-subgroup and K a Sylow q-subgroup of G. Clearly $H \cap K = 1$.

III.1. Show that $G = HK = KH$.

III.2. Show that if p does not divide $q - 1$, then H is normal.

From now on we assume that $q < p$.

III.3. Show that H is normal.

III.4. Show that if q does not divide $p - 1$ then G is abelian. (**Hint:** Use III.1 and I).

For $k \in K$, let k^* denote the function from H into H given by $k^*(h) = khk^{-1}$.

III.5. Show that for all $k \in K$, $k^* \in \mathrm{Aut}(H)$.

III.6. Show that the map * is a homomorphism from K into $\mathrm{Aut}(H)$.

III.7. Show that the map * determines G up to isomorphism.

III.8. Show that, up to isomorphism, there are at most $(p, q - 1)$ groups of order pq.

IV. Let G be a group.

IV.1. Show that if $G/Z(G)$ is cyclic, then G is abelian.

IV.2. For $x \in G$, show that there is a one-to-one correspondance between the conjugacy class x^G and the right coset space $G/C_G(x)$.

IV.3. Suppose G is finite. Show that

$$|G| = |Z(G)| + \sum_{x \in X} |G / C_G(x)|$$

for some subset X of $G \setminus Z(G)$.

IV.4. Conclude that a finite nontrivial p-group has a nontrivial center.

IV.5. Conclude that a group of order p^2 is abelian.

IV.6. Classify all groups of order ≤ 7.

V. Let G be an abelian group and H a divisible subgroup of G. Show that

$$G = H \oplus K$$

for some subgroup K of G.