Math 111
(Group Theory)

Resit II
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Ali Nesin

G will always denote a group, p stands for a prime natural number, n is a natural number. If $A \subseteq G$ and $H \leq G$, the **centralizer** of A in H is the subgroup

$$C_H(A) = \{h \in H : ha = ah \text{ for all } a \in A\},$$

the **center** of H is

$$Z(H) = C_H(H) = \{z \in H : hz = zh \text{ for all } h \in H\},$$

the **normalizer** of H in G is

$$N_G(H) = \{g \in G : gH = Hg\}.$$

$C_H(A)$, $Z(H)$ and $N_G(H)$ are all subgroups of G.

The subgroup H of G is called n-**divisible** if for any $h \in H$ there is an $x \in H$ such that $x^n = h$.

G is called **torsion-free** if for all $g \in G$ and $n > 0$, $g^n = 1$ implies $g = 1$.

1a. Show that if G is finite and has an element of order n, then n divides $|G|$. (5 pts.)

1b. Is the converse true? I.e. if $|G|$ is finite and divisible by n, is it true that G has an element of order n? (2 pts.)

2. Show that if $|G| = p$ (a prime) then G is cyclic. (5 pts.)

3a. What is the order of the element (1 2 3)(4 5 6)(7 8 9) in Sym(9)? (2 pts.)

3b. What is the subgroup of Sym(12) generated by the element (1 2)(3 4 5)(6 7 8 9)? (2 pts.)

4a. Show that the subgroups generated by $\sqrt{2}$ and $\sqrt{3}$ in $\mathbb{R}^+$ are isomorphic. (3 pts.)

4b. Show that the subgroup generated by $\sqrt{2}$ and $\sqrt{3}$ in $\mathbb{R}^*$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. (4 pts.)

5. Let $h \in G$ and A a finite subset of G be such that $A^h \subseteq A$. Show that $A^h = A$. (3 pts.)

6. Let φ be the homomorphism $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by

$$\varphi(x, y, z, t) = (x - y + t, z + t).$$

6a. Show that $\mathrm{Ker}(\varphi) \approx \mathbb{R} \times \mathbb{R}$. (7 pts.)

6b. Is φ onto? (3 pts.)

7. Show that if $H \triangleleft G$, then $[G, H] \leq H$. (4 pts.)

8. Let $g \in G$, $H \leq G$ and n and m two integers prime to each other.

8a. Assume that g^m, $g^n \in H$. Show that $g \in H$. (5 pts.)

8b. Assume that $g^m = g^n = 1$. Show that $g = 1$. (2 pts.)

9a. Show that if $A \subseteq B \subseteq G$, then $C_G(B) \leq C_G(A)$. (2 pts.)

9b. Show that for any $A \subseteq G$, $A \subseteq C_G(C_G(A))$. (3 pts.)

9c. Show that for any $A \subseteq G$, $C_G(A) = C_G(C_G(C_G(A)))$. (5 pts.)

10. Find a nonabelian group of order 8. (6 pts.)

11. Let $H = \{(a + b, a - b, a) : a, b \in \mathbb{R}\} \leq \mathbb{R}^3$. Show that $\mathbb{R}^3/H \approx \mathbb{R}$. (7 pts.)

12a. Show that for any $a \in G$, $a \in Z(C_G(a))$. (2 pts.)

12b. Show that if G is finite abelian and if $(n, |G|) = 1$ then G is n-divisible. (7 pts.)

12c. Show that if G is finite and if $(n, |G|) = 1$ then G is n-divisible. (**Hint:** Use parts a and b). (5 pts.)

13a. Show that $\mathbb{R}^+/\mathbb{Q}^+$ is torsion-free. (2 pts.)

13b. What is the cardinality of the set of elements of order 2 of $\mathbb{R}^*/\mathbb{Q}^*$. (4 pts.)

13c. Show that $\mathbb{R}/\mathbb{Z} \approx \mathbb{R}/2\mathbb{Z}$. (6 pts.)

13d. Find all subgroups of finite order of $\mathbb{Q}^*$. (4 pts.)

14. Show that $\langle g^2 : g \in G\rangle \lhd G$. (4 pts.)

15a. Let A and B be two finite cyclic groups of orders prime to each other. Show that $A \times B$ is cyclic. (6 pts.)

15b. Let A and B be two finite cyclic groups of orders **not** prime to each other. Show that $A \times B$ is not cyclic. (8 pts.)

16a. Show that for all $a \in G$, $C_G(a) = C_G(a^{-1})$. (4 pts.)

16b. Show that for all $a, x \in G$, $C_G(a)^x = C_G(a^x)$. (3 pts.)

16c. Let i and j be two elements of order 2 of G. Let $x = ij$. Show that $(ij)^i = (ij)^{-1}$. (3 pts.)

16d. Let i and j be two elements of order 2 of G. Show that $i \in N_G(C_G(ij))$. (4 pts.)

17. Show that any group G has maximal abelian subgroups. (10 pts.)