Group Theory

Homework
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Throughout G denotes a group.

1. Let G be such that $G/Z(G)$ is cyclic. Show that G is abelian.

2. Let $H \leq G$ and $g \in G$. Show that gHg^{-1} is a subgroup of G isomorphic to H.

3. Let $H, K \leq G$. Assume that for all $k \in K$, $kHk^{-1} \subseteq H$. Show that $K \leq \mathrm{N}_G(H)$.

4. The conjugacy class of an element $a \in G$ is defined to be

$$a^G = \{a^g : g \in G\}.$$

Note that $a \in a^G$.

4a. Show that two conjugacy classes are either disjoint or equal. In other words conjugacy classes partition G.

4b. Show that $|\mathrm{C}_G(a)| = 1$ iff $a \in Z(G)$ iff $|a^G| = 1$

4c. Let $G/\mathrm{C}_G(a)$ denote the right coset space $\{\mathrm{C}_G(a)g : g \in G\}$. Show that there is a one-to-one correspondance between $G/\mathrm{C}_G(a)$ and a^G.

4d. Assume now that G is finite. By using parts a, b and c show that

$$|G| = |Z(G)| + \sum |G/C_G(a)|$$

where the summation runs over a set of elements a of G such that $\mathrm{C}_G(a) \neq G$.

4e. Assume $|G| = p^n$ for some prime p and integer $n > 0$. By using part d show that $\mathrm{Z}(G) \neq 1$.

4f. Assume p is a prime and divides $|G|$. Show that G has an element of order p. (**Hint:** Proceed by induction on $|G|$. First consider the case when G is abelian. Then use part d).

5. Let $H \triangleleft G$ and $X \leq G/H$. For $g \in G$, let $\overline{g}$ denote the image of g in G/H. Let $K = \{g \in G : \overline{g} \in X\}$.

5a. Show that $K \leq G$.

5b. Show that $H \leq K$.

5c. Show that $X = K/H$.

5d. Show that $X \triangleleft G/H$ if and only if $K \triangleleft G$.

5e. Show that K is the unique subgroup of G containing H that satisfies part c.

6. Let $S = \{g^2 : g \in G\}$.

6a. Is S necessarily a subgroup? Prove or disprove.

6b. Show that $G'S$ is a normal subgroup of G. (G' denotes the subgroup generated by $\{x^{-1}y^{-1}xy : x, y \in G\}$).

6c. Show that $G/G'S$ is a group of exponent 2.

7. Let $H \triangleleft G$ and K a normal subgroup of G containing H.

7a. Show that $K/H \triangleleft G/H$.

7b. Show that $(G/H)/(K/H) \approx G/K$.

8. Let $H \lhd G$ and $K \leq G$.

8a. Show that $H \cap K \lhd K$.

8b. By a previous homework we know that $\langle H, K\rangle = HK$. Show that $HK/H \approx K/(H \cap K)$.

8c. Show that if H and K are two normal subgroups of finite index of G, then $H \cap K$ has also finite index in G.

9. Let $H \leq G$. Let $N_G(H) = \{g \in G : gH = Hg\}$.

9a. Show that there exists a one to one correspondance between the right coset space $G/N_G(H)$ and the set $\{H^g : g \in G\}$.

9b. Let $n = [G : H] < \infty$. Show that $|\{H^g : g \in G\}|$ divides n.

9c. Assume H has finite index in G. Using 7b and 6c, show that $\bigcap_{g\in G} g^{-1}Hg$ has finite index in G.

10a. Show that $\mathbb{Q}^+$ does not have a proper subgroup of finite index.

10b. Show that $\mathbb{Q}^+$ does not have a maximal proper subgroup.

10c. Show that among the subgroups of $\mathbb{Q}^+$ that does not contain 1, there is one which is maximal.

11. Let $H \leq G$. Show that there is a subgroup of G which is maximal with respect to the property $X \cap H = \{1\}$.

12. Let $H \leq G$. Show that $\bigcap_{g\in G} g^{-1}Hg$ is the unique maximal G-normal subgroup of H.

13. Let $H \leq G$. Let G/H denote the left coset space. For $g \in G$ and $xH \in G/H$, let $g^*(xH) = gxH$.

13a. Show that $g^* \in \mathrm{Sym}(G/H)$.

13b. Show that the map $g \to g^*$ is a homomorphism from G into $\mathrm{Sym}(G/H)$.

13c. What is the kernel of the homomorphism *?

13d. Assuming that $[G : H] = n < \infty$, show that $[G : \bigcap_{g\in G} g^{-1}Hg\ ]$ divides $n!$.

14. Let $G^{o} = G^{(o)} = G$ and for $i \in \mathbb{N}$, define $G^{i+1} = [G, G^{i}]$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

14a. Show that $(G^{i})_i$ and $(G^{(i)})_i$ are descending chain of normal subgroups of G.

14b. Show that $G^{(i)} \leq G^{i}$ for all i.

14c. Show that for $i \leq j$, $(G/G^{(j)})^{(i)} = (G^{(i)}/G^{(j)})$.

15. An element σ of $\mathrm{Sym}(n)$ is said to be of type $(n_1, ..., n_k)$ if $n_1 \leq ... \leq n_k$ and σ is the product of k disjoint cycles each of length $n_1, ... , n_k$. For example, the element

$$(1\ 2\ 3)(4\ 5)(6\ 7\ 8\ 9)(10\ 11)$$

is of type (2, 2, 3, 4).

15a. Show that two elements of Sym(n) are conjugate by an element of Sym(n) if and only if they have the same type. (Two elements σ and τ of a group G are said to be conjugate, if there is an $\alpha \in G$ such that $\sigma = \alpha^{-1}\tau\alpha$).

15b. Let $\sigma = (123)(45)$. Calculate $|\sigma^{\mathrm{Sym}(5)}|$ and $|\sigma^{\mathrm{Sym}(6)}|$.

15c. Compute $|C_{\mathrm{Sym}(5)}(\sigma)|$ and $|C_{\mathrm{Sym}(6)}(\sigma)|$.

15d. Find the elements of $C_{\mathrm{Sym}(5)}(\sigma)$ and $C_{\mathrm{Sym}(6)}(\sigma)$.